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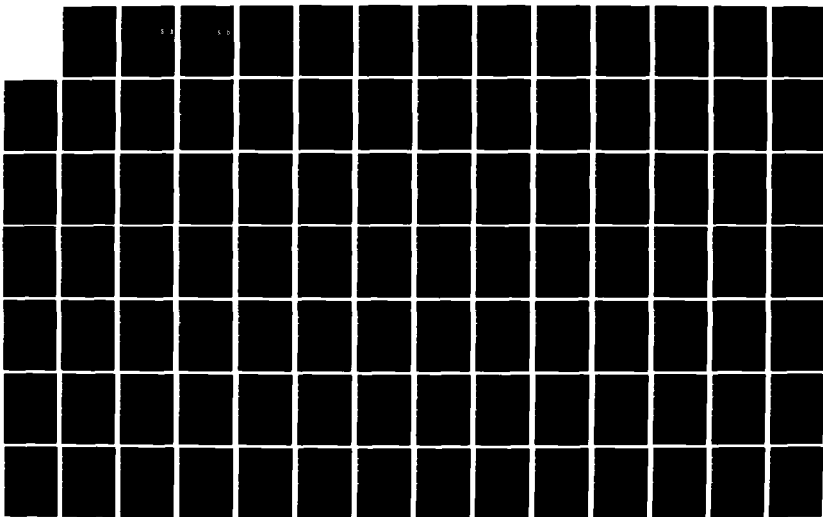
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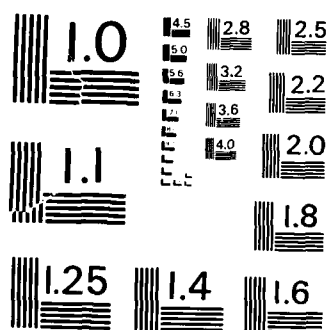
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SOLUTIONS OF
NONLINEAR MATRIX EQUATIONS

Thesis

Bruce W. Colletti
Captain, USAF

AFIT/GOR/MA/85D-1

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**SOLUTIONS OF
NONLINEAR MATRIX EQUATIONS**

Thesis

**Presented to the Faculty of the School of Engineering
of the Air Force Institute of Technology
Air University
in Partial Fulfillment of the
Requirements for the Degree of
Master of Science**

by

Bruce W. Colletti, B.S.

Captain, USAF

December 1985



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Preface

My earliest memory of mathematics recalls writing numbers in the spaces of a grid. Little did the first grader know that by creating this "box of numbers", he was laying the groundwork for subsequent study in matrices, themselves very special "boxes of numbers." After studying matrix theory as a high school freshman, I remember hoping for a chance someday to pursue it in great depth. Though the hope has now been realized, it will by no means be put to rest: for my master's in mathematics, I will pick up where this thesis left off.

Fortune doesn't always favor us, and as Machiavelli says, will strike after being wooed and cajoled. However, she must have been in a good mood in the Fall of '84 because our class had Dr. John Jones Jr for Numerical Analysis. His presentation of matrix algebraic concepts delighted me, and upon discovering his abiding interest and renown with matrix equations, decided to study under him. Undoubtedly, Dr Jones was one of two teachers having a profound impact on my mathematical development. I couldn't have wished for a finer advisor and coach than him. His patience and positive outlook sustained me during those times when I was setting back mathematical science instead of advancing it. For him, theory and applications are one, though primacy goes to theory. Just as it should be. Thanks for your interest in me, Dr Jones. I will return your concern years hence to one

of my students.

To LTC (and Dr.) Charles Ebeling, my reader in the Operations Research Department, I want to thank for being (I believe) a mathematician at heart, and hence an ally in the department. If I couldn't have done a mathematical thesis, I would have done a simulation one and sought you out as an advisor. Thanks for not discouraging me from pursuing an interest begun years ago.

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List of Symbols

$0_{m \times n}$	an $m \times n$ matrix whose entries are all zero
$A \triangle B$	tensor product of matrices A and B
$A_{m \times n}$	the $m \times n$ matrix A
\mathbb{C}	the set of complex numbers
$\mathbb{C}(\Gamma)$	the set of polynomials with coefficients in the complex numbers and variables in the components of the n -tuple Γ
$\mathbb{C}^{m \times n}(\Gamma)$	the set of $m \times n$ matrices whose entries are polynomials with coefficients in the complex numbers and variables in the components of the n -tuple Γ
$\mathbb{C}(x)$	the set of polynomials in the variable x
$\mathbb{C}(x, y)$	the set of polynomials in the variables x, y
$\det(A)$	the determinant of the matrix A
iff	if and only if
$I_{n \times n}$	the $n \times n$ identity matrix
$\text{rank}(A)$	the rank of the matrix A
$S_1 - S_2$	the set of elements which are in the set S_1 but not in the set S_2
Γ	an arbitrary n -tuple $(\Gamma_1, \dots, \Gamma_n)$
$x \in S$	the entity x is contained in the set S
\mathbb{Z}^+	the set of positive integers ≥ 0

Abstract

This paper seeks the solutions to a system of equations (equalities) in n variables by expressing the system in matrix algebraic form. Properties of the solutions to the ensuing matrix equation are investigated using similarity transformations. The three types of matrix equations to be studied are the linear equation -

$$AX = b$$

the Lyapunov equation -

$$AX - XB = C$$

the second-order Riccati equation -

$$XDX + AX + XB + C = 0$$

and the third-order Riccati equation -

$$XAXBX + XCX + DX + XE + F = 0$$

The entries of all matrices, including the solution X , are restricted to being polynomials in \mathbf{r} having complex coefficients, where \mathbf{r} is the n -tuple of indeterminates. That is, all matrices are elements of the ring $\mathbb{C}^{m \times n}(\mathbf{r})$ for m and n of appropriate size.

Because adding and multiplying matrices (having multivariate polynomial entries) is tedious in practice, an interactive BASIC program is presented in the appendix. This program, which can be used on a personal computer, permits the user to perform operations on matrices having multivariate polynomial entries. Via menu selections, the user may perform

-weighted addition between two matrices

- multiplication between two matrices
- create matrices, with an option of building a diagonal matrix whose diagonal entries are all equal
- view matrices
- transpose a matrix
- extract special submatrices (U,M,V,N of Chapter IV) from a given matrix.

I Overview and Literature Review

Overview. This paper seeks the solutions of the third-order Riccati matrix equation

$$XAXBX + XCX + DX + XE + F = 0 \quad (1.1)$$

where the entries of the matrices A,B,C,D,E,F,X are multivariate polynomials. In solving (1.1) the linear matrix equation

$$AX = b$$

the Lypanov matrix equation

$$AX - XB = C$$

and the second-order Riccati matrix equation

$$XDX + AX + XB + C = 0$$

will be addressed because the form of their solutions will hint at the nature of those to (1.1). Because cursory knowledge of matrix algebra is sufficient to motivate the paper's thesis--the solution to (1.1)--the rigorous (and lengthy) definition of terms and statement of objective will be made in the next chapter.

Why a matrix equation (such as those given above) is worthy of attention, let alone finding its solution, is a legitimate concern. Practical problems often arise which require the solution to a system of m equations in n unknowns, e.g., the system of equations

$$\begin{aligned} ax + by &= c \\ dx + ey &= f \end{aligned} \quad (1.2)$$

where all variables except x and y are known. One way to

find the solutions x and y is to express the system in its equivalent matrix form

$$\begin{bmatrix} a & b \\ d & e \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c \\ f \end{bmatrix}$$

(which is a linear matrix equation) and solve for x and y , i.e.

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a & b \\ d & e \end{bmatrix}^{-1} \begin{bmatrix} c \\ f \end{bmatrix}$$

Thus, a system of equations may give rise to a matrix equation whose solution in turn gives the answer to the original system.

Other more complex systems may have a matrix representation. For instance

$$\begin{aligned} \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} e \end{bmatrix} & (1.3) \\ &= \begin{bmatrix} xa + yc & xb + yd \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \\ &= \begin{bmatrix} x(xa + yc) + y(xb + yd) \end{bmatrix} \\ &= \begin{bmatrix} x^2a + xy(c + b) + y^2d \end{bmatrix} \end{aligned}$$

which means that the system consisting of the one equation

$$x^2a + xy(c + b) + y^2d = e$$

is represented by the matrix equation (1.3). These two examples show that matrix equations warrant attention if for no other reason than they can help solve systems of equations arising in practical problems. Indeed, given the nonlinear optimization challenge

$$\begin{array}{l} \text{Maximize } f(x) \text{ subject to} \\ \text{equality constraint,} \\ \cdot \\ \text{equality constraint,} \end{array} \quad (1.4)$$

the constraint set may be expressable as a matrix equation. If so, this equation can be solved to identify the feasible region determined by the constraints and thence the optima. This approach is an alternative to solving (1.4) using, say, Lagrangian multipliers and partial differential calculus.

In examples (1.2) and (1.4), the variables are often taken to represent real or complex numbers. However, there is nothing stopping the variables from assuming functional values. For instance (1.2) may assume the form

$$\begin{array}{l} a(u,v)x + b(u,v)y = c(u,v) \\ d(u,v)x + e(u,v)y = f(u,v) \end{array} \quad (1.5)$$

where a, b, c, d, e, f are functions of the parameters u and v . Though the solutions x and y of (1.5) wouldn't necessarily have numeric values, this is all right because all other variables are functions: x and y will likely take on functional forms. To illustrate,

$$\begin{aligned} \begin{bmatrix} u^2v & u \\ uv & v \end{bmatrix} \begin{bmatrix} u+v \\ u-v \end{bmatrix} &= \begin{bmatrix} u^2v(u+v) + u(u-v) \\ uv(u+v) + v(u-v) \end{bmatrix} \\ &= \begin{bmatrix} u^3v + u^2v^2 + u^2 - uv \\ u^2v + uv^2 + uv - v^2 \end{bmatrix} \end{aligned}$$

and so a solution to the matrix equation

$$\begin{bmatrix} u^2v & u \\ uv & v \end{bmatrix} \begin{bmatrix} x(u,v) \\ y(u,v) \end{bmatrix} = \begin{bmatrix} u^3v + u^2v^2 + u^2 - uv \\ u^2v + uv^2 + uv - v^2 \end{bmatrix}$$

is $x(u,v) = u + v$ and $y(u,v) = u - v$. As given in (1.1), this paper will address matrix equations whose entries

assume the functional form of a polynomial in several variables.

Another area where matrix equations appear (where the matrices' entries are multivariate functions) is control theory, the science addressing the orbital natures of objects in space. Matrix equations frequently arise from systems of differential equations which describe a satellite's orbit. For example, given $X(t)$ a columnar matrix whose entries (each describing an aspect of a satellite's orbit) are functions in the parameter t , a theorem from control theory states that the equilibrium of the system of differential equations

$$\frac{dX}{dt} = FX$$

is asymptotically stable if the Lypanov matrix equation

$$F^T P + P F = -C \quad (1.6)$$

has a positive-definite solution P for any matrix $C > 0$. Though the terms in this theorem won't be defined here (see [4:144-152]), the important item is that a satellite's orbital stability depends on the solution of a Lypanov matrix equation (1.6).

Matrix equations can also describe the path taken by an X-ray passing through matter. This is a concern of the medical community since the quality of CAT scan pictures depends on the manner in which the X-rays penetrate the skull. A good description of the path may allow finding a

brain tumor, whereas a poorly chosen angle of entry may not. R. Vasudevan [32] develops higher order Riccati equations describing how beam particles scatter upon hitting matter, e.g. the skull. Simply put, scattering is modelled by including higher powers of a solution matrix X in a Riccati equation. For instance, scattering may be roughly described by the second order Riccati equation

$$XDX + AX + XB + C = 0$$

and more completely by the third order Riccati equation

$$XAXBX + XCX + DX + XE + F = 0$$

where the coefficient matrices A, B, C, D, E, F reflect known characteristics of the environment in which the X-rays behave. Higher powers of X will more accurately describe scattering. Bellman and Vasudevan [2] describe techniques reducing a given Riccati equation to one of lower form. This reduction (quasi-linearization) leads to a series which converges to the actual solution of the original equation.

Literature Review. The literature supporting this paper addresses three topics:

- I generalized inverses of matrices
- II algebraic structures of matrices over fields and rings
- III the Lypanov and Riccati equations

Category I. An understanding of matrix generalized inverses launched the research behind this paper. Matrix generalized inverses encompass the traditional notion of an inverse by assigning these to non-square matrices. The theory is well developed for

matrices whose entries are complex numbers, and is described below.

The physicist R. Penrose [28] proved in 1955 that for any matrix A whose entries are complex numbers, there exists matrices U, Y, Z, W having complex entries such that:

1. $AUA = A$ (1.7)
2. $YAY = Y$
3. AZ equals its conjugate transpose (i.e., AZ is Hermitian)
4. WA is Hermitian

The matrices U, Y, Z, W are known as the "generalized inverses" of A . Penrose also showed the existence of a matrix X which simultaneously satisfied conditions (1) thru (3), and a Y simultaneously satisfying (1), (2) and (4). Penrose also showed that if a matrix X satisfied all conditions (given a matrix A), then X was unique. From the traditional matrix algebra viewpoint, this unique X corresponds to the familiar inverse of a square matrix A whose determinant is not equal to zero. Though Penrose's work was original, he was unaware that the mathematician E.H. Moore [24] had proved (using abstract algebra) the existence of generalized inverses for arbitrary rings more than three decades earlier.

Rao and Mitra's textbook [29] is devoted to the study of generalized inverses, and is a frequently cited authority in this area. The first two chapters of [6] present techniques (with the justifying theorems) generating various types of generalized inverses for a given matrix A , while Captain Craig Murray, AFIT Class GCS 85D, has recently

completed the programming of these techniques. Two papers, [12] and [21], use generalized inverses to prove theorems about special types of matrix equations whose entries are complex numbers.

The power of generalized inverses weakens as one moves from matrices with complex entries to those with multivariate entries, the subject of this paper. This is because the theory in the latter is still very young. Sontag [31] discusses the existence of some generalized inverses for special matrices, while Jones [11] uses these generalized inverses to solve certain matrix equations.

Category II. Most of this paper's research dealt with the decisive influence on the existence and form of solutions to matrix equations: the algebraic structure of matrices. An illustration of the importance of algebraic structure is readily given: the solution of the equation

$$x + 7 = 5 \tag{1.8}$$

cannot be found among the set of positive integers Z^+ (even though the coefficients 7 and 5 are positive integers) because its structure doesn't include negative numbers. Since -2 is the solution, (1.8) must be placed in a larger algebraic structure in order for a solution to exist. Thus one moves from Z^+ to the set of all integers Z . Though Z has a larger algebraic structure than Z^+ it still isn't large enough to contain the solution of the equation

$$2x - 8 = 7 \tag{1.9}$$

even though the coefficients 2,-8,7 are found in \mathbb{Z} . It now becomes necessary to move to a larger structure (the set of rational numbers) to find the solution to (1.9), $15/2$.

Likewise, to solve the equations $x^2 = 2$ and $x^2 + 1 = 0$ one must move to the larger algebraic structures found in the irrational and complex numbers respectively.

Matrices too lend themselves to different structures. However, not only must the form of a matrix be addressed (e.g., whether it's invertible, whether it's similar to another matrix), but the structure in which individual entries are found must also be considered. For instance, decomposing a matrix into a product of other matrices may depend upon whether or not an entry is irreducible in its own setting. A trivial example is decomposing the 1×1 matrix

$$[x^2 + x - 1] \quad (1.10)$$

into a product of two other 1×1 matrices each having polynomial entries whose coefficients are integer, as is the entry in (1.10). Such a decomposition is impossible because $x^2 + x - 1$ cannot be expressed as a product of two polynomials with integer coefficients. Indeed, the roots of this equation are $(-1 \pm \sqrt{5}) / 2$, which are irrational.

In general, this paper allows matrix entries to be multivariate polynomials (having complex coefficients, e.g., $x^2 + xy$, $xy^2z - (2 + 3i)x^2yz$), who in turn have complex algebraic structures. In fact, a solid understanding of

abstract algebra would have helped make several articles intelligible. Though Fraleigh's text [7] was found to be an excellent primer on abstract algebra, time did not allow sufficient study of a subject key to this paper. Wang [33] gives an algorithm for irreducible factoring of multivariate polynomials having coefficients in an arbitrary algebraic number field. McClellan [23] presents methods for solving systems of equations involving univariate polynomials with rational coefficients (this paper highlights his doctoral dissertation). Two papers from the early twentieth century, [3] and [27], discuss the form of factors between polynomials.

The algebraic structures of matrices was addressed by Frost and Storey [8] and Lee and Zak [20]. These special structures, called Smith Forms, are discussed in the next chapter. Unfortunately, only the Smith Forms of matrices having bivariate polynomial entries can be addressed: the Smith Form involving multivariate entries remains an unresolved issue in mathematics.

Category III. Working from a base rooted in generalized inverses and recognizing the critical role of algebraic structures on matrices (and again among their entries), a study of matrix equations can proceed. Though much has been published on finding solutions to systems of equations, many authors succeed in finding only particular solutions. After all, finding the general forms of all

solutions may have either been a far too ponderous task or the mathematical approach proved elusive. In any case, viewing a system of multivariate equations as a single matrix equation (having a solution in its own right) is not commonplace. Depending on the nature of the system, the entire solution set may be found by solving a matrix equation representing the system.

Work has been done on matrix equations having complex entries. Roth [30] (a frequently cited paper) wrote on the Lyapunov matrix equation in 1952, and had his work extended in 1972 by Jones [12] who also addressed the second order Riccati equation, using generalized inverses to identify solutions. Morris and Odell [25] attempted to find the common solutions to a set of linear matrix equations $\{A_k X = B_k\}$ by using generalized inverses. Lancaster [18] provides several approaches (none using generalized inverses) to solving the matrix equation

$$\sum A_k X B_k = C$$

As matrix equations assume multivariate polynomial entries, the constraints cited in the previous section seriously handicap the search for solutions. As a result, pioneering work in these equations is still underway. The papers [11], [16] and [17] give recent results connected with the Lyapunov and second/third order Riccati equations. Collectively, these papers address the roles of generalized inverses and matrix algebraic structures in the search for

solutions to these matrix equations.

II Foundations

This chapter lays the groundwork for later developments. Because this paper seeks solutions to matrix equations, the latter's nature will be addressed first.

Matrix Equations. A matrix equation is an algebraic expression whose arguments involve matrices. Matrix equations are frequently used to represent a kindred system of equations. For instance, the system of equations

$$\begin{aligned} 2x + 3y &= 7 \\ 4x - 2y &= 1 \end{aligned}$$

is represented by the matrix equation

$$\begin{bmatrix} 2 & 3 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ 1 \end{bmatrix}$$

which has the familiar form

$$AX = b \tag{2.1}$$

where

$$A = \begin{bmatrix} 2 & 3 \\ 4 & -2 \end{bmatrix} \quad X = \begin{bmatrix} x \\ y \end{bmatrix} \quad b = \begin{bmatrix} 7 \\ 1 \end{bmatrix}$$

The reason for expressing a system of equations in matrix algebraic form is, of course, so that the machinery of matrix algebra can be used to find a solution to the given initial system.

Other systems of equations have matrix representations. For instance, the system

$$\begin{aligned} x + 2u &= 5 \\ 3x + 4u &= 7 \\ 9x + 10u &= 23 \\ y + 2v &= 6 \\ 3y + 4v &= 8 \\ 9y + 10v &= 24 \end{aligned}$$

2.1

has the matrix form

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 9 & 10 \end{bmatrix} \begin{bmatrix} x & y \\ u & v \end{bmatrix} = \begin{bmatrix} 5 & 6 \\ 7 & 8 \\ 23 & 24 \end{bmatrix}$$

which in turn has the form (2.1) where

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 9 & 10 \end{bmatrix} \quad X = \begin{bmatrix} x & y \\ u & v \end{bmatrix} \quad b = \begin{bmatrix} 5 & 6 \\ 7 & 8 \\ 23 & 24 \end{bmatrix}$$

Thus in viewing (2.1), it isn't necessary that b and X be one-columnar matrices and A square, as is traditionally presented.

Lypanov Equation. The linear matrix equation (2.1) leads to the slightly more complex equation

$$AX + XB = C \quad (2.2)$$

which is known as the Lypanov equation. In order for (2.2) to be defined, the matrices A, B must be square (though not necessarily of the same size) and the matrices X, C must share the same size (though they may be rectangular). To see why this is true, let the following hold:

MATRIX	A	X	B
SIZE	$a \times b$	$c \times d$	$e \times f$

For AX to be defined, b must equal c : $b = c$. For XB to be defined, $d = e$. AX is $a \times d$ and XB is $c \times f$. For $AX + XB$ to be defined, $a = c$ and $d = f$. Since $b = c$ and $a = c$, $a = b$ and A is square $c \times c$. Similarly for B : $d = e$ and $d = f$ and so $e = f$ which makes B square $d \times d$. It follows that $AX + XB$ is $c \times d$ which is the size of X .

To illustrate (2.2), the system

$$\begin{aligned}
 (x + 2u) + (5x + 8y + 11z) &= 14 \\
 (y + 2v) + (6x + 9y + 12z) &= 15 \\
 (z + 2w) + (7x + 10y + 13z) &= 16 \\
 (3x + 4u) + (5u + 8v + 11w) &= 17 \\
 (3y + 4v) + (6u + 9v + 12w) &= 18 \\
 (3z + 4w) + (7u + 10v + 13w) &= 19
 \end{aligned} \tag{2.3}$$

has the Lypanov form

$$\begin{aligned}
 &\begin{matrix} (A) & X \\ \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} & \begin{bmatrix} x & y & z \\ u & v & w \end{bmatrix} \end{matrix} + \begin{matrix} X \\ \begin{bmatrix} x & y & z \\ u & v & w \end{bmatrix} \end{matrix} \begin{matrix} (B) \\ \begin{bmatrix} 5 & 6 & 7 \\ 8 & 9 & 10 \\ 11 & 12 & 13 \end{bmatrix} \end{matrix} \\
 &= \begin{matrix} (C) \\ \begin{bmatrix} 14 & 15 & 16 \\ 17 & 18 & 19 \end{bmatrix} \end{matrix}
 \end{aligned}$$

Of course, (2.3) can be expressed as (2.1) by combining like terms, thereby creating six equations in six unknowns.

However, it will be seen (Chapter IV) that by solving the Lypanov equation instead of its equivalent linear form

(2.1), solutions of a higher order matrix equation can be derived, one in which the Lypanov is embedded. This higher equation is known as the second-order Riccati equation.

Riccati Equation and Higher. In high school algebra, a student moves from solving the simple linear equation $bx = c$ to the higher order quadratic equation $ax^2 + bx = c$ (in which is embedded the linear equation $bx = c$). In the present setting, the former equation is likened to the Lypanov matrix equation and the second-order Riccati matrix equation to the quadratic. The Riccati equation has the form

$$XDX + AX + XB + C = 0 \tag{2.4}$$

For (2.4) to be defined, A and B must be square (possibly

not the same size), D must have the size of X^T , and X, C share the same size. The argument for this follows that of the Lyapunov equation.

To illustrate (2.4), the Riccati form of the system

$$\begin{aligned} x^2 + uy + 2x &= -1 \\ xy + yv + 2y &= -2 \\ xz + yw + 2z &= -3 \\ ux + vu + 2u &= -4 \\ uy + v^2 + 2v &= -5 \\ uz + vw + 2w &= -6 \end{aligned} \quad (2.5)$$

is

$$\begin{aligned} & \begin{pmatrix} X \\ x & y & z \\ u & v & w \end{pmatrix} \begin{pmatrix} D \\ \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \end{pmatrix} \begin{pmatrix} X \\ x & y & z \\ u & v & w \end{pmatrix} + \begin{pmatrix} A \\ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{pmatrix} \begin{pmatrix} X \\ x & y & z \\ u & v & w \end{pmatrix} = \\ & \begin{pmatrix} X \\ x & y & z \\ u & v & w \end{pmatrix} \begin{pmatrix} B \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{pmatrix} + \begin{pmatrix} D \\ \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{aligned}$$

As with the Lyapunov equation, there may exist many solutions to the second-order Riccati equation. In fact it will be seen in Chapters III and IV that the solutions of (2.4) will satisfy pairs of equations determined by the matrices A, B, C, D in (2.4).

As it is an easy matter to create a polynomial of a given degree (e.g., $3x^3 + 3x^2 - 7$), so is it also to create a higher order matrix equation. For instance, the third order Riccati equation

$$XAXBX + XDX + AX + XB + C = 0 \quad (2.6)$$

or the related matrix equation

$$EXAXBX + GXDXH + AXJ + KXB + C = 0$$

where the matrices not equal to X are known and so can be

thought of as coefficients of a polynomial in x . However, the non-commutativity of matrix multiplication gives the placement of each coefficient matrix a decisive influence on the solution.

A natural question that arises is whether or not a given system has an accompanying matrix form. At present, the only known way to answer this question is to experiment, an understandably unattractive task. However, if a matrix representation is derived (or stumbled upon), the concepts presented in this paper will help to identify the solutions to the given matrix equation, and thus to its underlying system of equations.

Matrices Over a Ring. Two items determine the nature of a matrix: its size and entries. Concerning the latter, a matrix is said to be "over a set S " if and only if its entries come from the set S . For example if $S = \{1,2,3\}$ then the 1×4 matrix

$$A = [3 \ 1 \ 1 \ 2]$$

is "over S " because each of its entries belong to S . However, the 2×3 matrix

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & \wedge \end{bmatrix} \quad (2.7)$$

is not "over S " because $B(2,3) = 0$, which is not a member of S . If however, $S = \{\text{the real numbers}\}$, then B is "over S ". Likewise, if

$$A = \begin{bmatrix} 2 + 3i & 7 \\ 9 & 8 - 7i \end{bmatrix} \quad (2.8)$$

then A is not a matrix over the real numbers, but is a matrix over the complex numbers. In this paper, the set of complex numbers will be given by the symbol \mathbb{C} (see the "List of Symbols").

These examples lead to a notation used throughout this paper: given a set S , the set of all $m \times n$ matrices whose entries are in S is symbolically given by

$$S^{m \times n} \quad (2.9)$$

If A is an $m \times n$ matrix whose entries are in S , then the statement " A is an element of $S^{m \times n}$ " is represented by

$$A \in S^{m \times n}$$

Example: From (2.8), $A \in \mathbb{C}^{2 \times 2}$ because A is a 2×2 matrix whose entries are complex. However from (2.7), B is not in $\mathbb{C}^{2 \times 2}$ because B is a 2×3 , despite the fact that the entries of B are elements of the complex numbers (the set of real numbers is a subset of the complex numbers). However, $B \in \mathbb{C}^{2 \times 3}$ ■

The concept of "overness" will now be extended. Let

$$S(x) = \{\text{all polynomials in the variable } x \\ \text{with coefficients in the set } S\} \quad (2.10)$$

Thus if $S = \{1, 2, 3\}$, then

$$f(x) = x^2 + 2x + 3 \in S(x)$$

because $f(x)$ is a polynomial in x with coefficients in S .

However

$$f(y) = y^2 + 2y + 3$$

is not in $S(x)$ because $f(y)$ is a polynomial in y . Likewise,

$$f(x) = 7x^2 + 2x + 3$$

is not in $S(x)$ because the coefficient of x^2 is seven, which is not found in $S = \{1, 2, 3\}$. It follows that $\Phi(x)$ is the set of polynomials (in the variable x) with complex coefficients.

Similar to (2.10), let

$$S(x, y) = \{\text{polynomials in the variables } x, y \text{ whose coefficients are in the set } S\}$$

Thus if $S = \{4, 5, 2\}$, then

$$f(x, y) = 4x^{10}y^5 + 2x^2y + 5x + 5y \in S(x, y)$$

because $f(x, y)$ is a polynomial in x, y with coefficients in S . It follows that $\Phi(x, y)$ is the set of polynomials in the variables x, y with complex coefficients.

In general, let \mathbf{n} represent the n -tuple (x_1, x_2, \dots, x_n) . Then given a set S , $S(\mathbf{n})$ is defined to be the set of all polynomials (in the variables found in \mathbf{n}) with coefficients in the set S .

Example: If $\mathbf{n} = (x, y, z, v, w)$, then $\Phi(\mathbf{n})$ is the set of all polynomials in the variables x, y, z, v, w with complex coefficients. Thus

$$f(x, y, z, v, w) = (4+5i)x^2 + xyzvw - (2+12i)w^7 \in \Phi(\mathbf{n}).$$

However if S is the set of real numbers, then $f(x, y, z, v, w)$ is not in $S(\mathbf{n})$. ■

Now to extend notation to matrices. Define

$$S^{m \times n}(\mathbf{n}) = \{m \times n \text{ matrices whose entries are in } S(\mathbf{n})\} \quad (2.11)$$

Thus if $S = \{1,2,4,8,9\}$ and $\mathfrak{n} = (x,y,z)$ and

$$A = \begin{bmatrix} x^2yz + 9z & x \\ 2x & 4yz + 8z^2 \end{bmatrix}$$

then $A \in S^{2 \times 2}(\mathfrak{n})$ because A is 2×2 , and each entry of A is a polynomial in $\mathfrak{n} = (x,y,z)$ with coefficients in S . However if $S = \{1,2,4,8\}$, then A would no longer be in $S^{2 \times 2}(\mathfrak{n})$ because a coefficient of $A(1,1)$ isn't in S (namely, 9). From (2.11), $\mathbb{C}^{m \times n}(\mathfrak{n})$ is the set of all $m \times n$ matrices whose entries are polynomials in \mathfrak{n} having complex coefficients.

With this final definition, the object of this paper can be stated: find the solution $X \in \mathbb{C}^{m \times n}(\mathfrak{n})$ of the third-order Riccati equation

$$XAXBX + XCX + DX + XE + F = 0 \quad (2.12)$$

given \mathfrak{n} the tuple of indeterminates in the polynomial entries of $A,B,C,D,E,F \in \mathbb{C}^{k \times j}(\mathfrak{n})$ for k,j,m,n of appropriate size.

One may feel that in (2.12), mandating $X \in \mathbb{C}(\mathfrak{n})$ is unduly restrictive. After all, forcing the solutions of $x^2 - 2 = 0$ (a polynomial in x over the integers, i.e., a polynomial with integer coefficients) to again be a polynomial over the integers isn't possible, since $\sqrt{2}$ is irrational. Also, given the matrix equation $AX = b$ with the entries of A,b integer and A square and invertible, it may not be possible to have the solution $X = A^{-1}b$ to again have integer entries. Why then require X in (2.12) to have entries in $\mathbb{C}(\mathfrak{n})$ when (like these two analogies) it may be

necessary to leave $\mathcal{Q}(n)$? The argument is well posed, since a given matrix equation may indeed not have a solution in a given space. In fact, practical solutions could be missed by restricting solutions for theoretical reasons.

The reasons for mandating $X \in \mathcal{Q}^{m \times n}(n)$ are in part pragmatic. First of all, polynomial entries don't lend themselves to singularities: X would be defined at all values of n . This is important because given an entry that is a rational function (e.g., $1/x$), there may exist a singularity or discontinuity at a value which in fact does have physical significance. Because of the rational entry, the form of the solution may prove untenable. Secondly, many functions can be approximated by a series of polynomials, e.g., Chebysev polynomials [5:239]. In fact, computational considerations may force the analyst to use polynomial approximations of a function. Restricting the entries of X to $\mathcal{Q}(n)$ would help generate these polynomials to be used in approximating the solutions to a phenomena.

The driving force, however, for requiring $X \in \mathcal{Q}^{m \times n}(n)$ is theoretical: is it possible to find some solutions in the same space as the coefficient matrices in a given matrix equation? If so, how would the solutions be obtained? It may very well be that solutions thus found may shed light on the nature of solutions which lie outside the required space.

A word needs to be said on the setting for addition and

multiplication on matrices over $\mathbb{Q}(n)$. Though these operations seem to hardly merit concern, there is in fact an extremely important algebraic structure behind them. Some of the current literature on matrix equations refers to this "ring" structure of $\mathbb{Q}^{m \times n}(n)$.

In the broad setting, a set of objects S (e.g., matrices) is given, and two operations, $+$ and $*$, are defined (e.g., matrix addition and multiplication) between a pair of elements of S . These operations have the property that if $a, b \in S$ then: $a + b$, $a * b \in S$ and $a + b$, $a * b$ assume unique values (e.g., $a + b$ has one value and one value only). Operations with these closure and uniqueness properties are called binary operations. Depending on the nature of the operations, an algebraic structure (denoted by $\langle S, +, * \rangle$) is defined on the set S taken together with the operations. For S the set of matrices over $\mathbb{Q}(n)$ and the customary operations of matrix addition and subtraction, $\langle S, +, * \rangle$ is known as a ring.

A ring $\langle R, +, * \rangle$ is a set R together with two binary operations $+$ and $*$ defined on R such that the following axioms are satisfied [7:195]:

- R1. $\langle R, + \rangle$ is an abelian group.
- R2. $*$ is associative.
- R3. For all $a, b, c \in R$,

$$a * (b + c) = (a * b) + (a * c)$$

$$\text{and}$$

$$(a + b) * c = (a * c) + (b * c)$$

It is an easy matter to confirm that the operations of matrix addition and multiplication satisfy the ring axioms.

Rank and Determinant of a Matrix. This chapter closes by addressing two familiar properties of a matrix: its determinant and rank. Both will be cited throughout this paper.

The rank of an $m \times n$ matrix A (denoted by $\text{rank}(A)$) is the number of linearly independent columns in A . Chapter III will show that $\text{rank}(A)$ affects the solution of the matrix equation $AX = b$, where A, X, b are matrices over $\mathbb{Q}(\mathbb{P})$. In finding $\text{rank}(A)$, the determinant of a matrix will come into play.

Despite the many techniques which find the determinant of a square matrix, later proofs will refer to the first historical definition of a determinant, computationally inefficient though it is. This definition, later referred to as the "permutational" form of a determinant, derives its name from its indexing (of matrix elements) on a permutation of a set [26:89].

Given the set of integers $S = \{1, 2, \dots, N\}$ a permutation on S is a one-to-one and onto mapping σ from S into itself, and is represented by a $2 \times N$ matrix whose top row is S and whose bottom row is a specific permutation of S . For instance, the two permutations of $\{1, 2\}$ are given by

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}$$

For $S = \{1, 2, 3, 4\}$ one permutation is given by

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 3 & 1 \end{pmatrix}$$

(where $\sigma(1) = 2$, $\sigma(2) = 4$, $\sigma(3) = 3$ and $\sigma(4) = 1$) and another by

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}$$

where $\sigma(1) = 4$, $\sigma(2) = 3$, $\sigma(3) = 2$ and $\sigma(4) = 1$.

If $A = [a_{kj}]$ is an $N \times N$ matrix, then the determinant of A is given by

$$\det(A) = \sum_{p \in S} [\operatorname{sgn}(p) \prod_{j=1}^N (a_{j,p(j)})] \quad (2.13)$$

where

S is the set of all permutations on $\{1, 2, \dots, N\}$

and

$\operatorname{sgn}(p) = \pm 1$, depending on the number of transpositions which comprise the particular permutation p [7:47-48].

III Solving the Linear Equation $AX = b$

The key to solving the Lyapunov and Riccati equations involves finding the solutions of the linear equation

$$AX = b \quad (3.1)$$

As before, matrices may be non-square but entries must be in $\mathbb{C}(\mathbb{R})$.

Though the matrices of (3.1) may be rectangular, one is first led to investigating properties of a square matrix $A \in \mathbb{C}^{n \times n}(\mathbb{R})$ —in particular, whether or not A is invertible. As in the real case, if $\det(A)$ equals zero, then A has no inverse. In general however, the determinant of A is a polynomial over \mathbb{R} and thus has an inverse except at the roots of $\det(A)$. For example, if

$$A = \begin{bmatrix} x + y & x \\ y & xy \end{bmatrix} \in \mathbb{C}^{2 \times 2}(x, y) \quad (3.2)$$

then

$$\det(A) = (x + y)xy - xy = xy(x + y - 1)$$

and A has an inverse except in the root set of $\det(A)$

$$\{(x, y) : x = 0 \text{ or } y = 0 \text{ or } x + y = 1\}$$

If A must be invertible for all values of \mathbb{R} (as will soon be required), then $\det(A)$ can only be a non-zero complex number. In this case A^{-1} is easily found using the well known formula [26:96]

$$A^{-1} = |A|^{-1} [\text{matrix-of-cofactors of } A]^T$$

That $A^{-1} \in \mathbb{C}^{n \times n}(\mathbb{R})$ can be seen by recalling that the cofactor of an entry a_{ij} of A is the determinant of a

3.1

submatrix of A (obtained by deleting the i 'th row and j 'th column of A). Because the entries of A are in $\mathbb{C}(n)$, the cofactors are also in $\mathbb{C}(n)$, and thus the transpose of the matrix of cofactors is in $\mathbb{C}^{n \times n}(n)$. Since $\det(A)$ is assumed to be a non-zero complex number, so too will $|\det(A)|^{-1}$ which implies $A^{-1} \in \mathbb{C}^{n \times n}(n)$. With this in mind, a specific form of (3.1) can be solved for X : if $A \in \mathbb{C}^{n \times n}(n)$ and $\det(A) \in \mathbb{C} - \{0\}$ then there exists an $A^{-1} \in \mathbb{C}^{n \times n}(n)$ and $X = A^{-1}b$ (the general setting for the familiar result involving real matrices). Finally, since the entries of A^{-1} and b are in $\mathbb{C}(n)$, so too are the entries in their product $X = A^{-1}b$ as mandated in (3.1).

The picture becomes more complicated when A doesn't have an inverse. This may be because $\det(A) = 0$, $\det(A) \in \mathbb{C}(n) - \mathbb{C}$, or $\det(A)$ isn't defined, i.e. A rectangular. Nevertheless, (3.1) may still have a solution and in fact, may have infinitely many. However, before the general solution of (3.1) can be derived, a more universal setting for 'inverses' of a matrix will be addressed.

The basic concept here involves elementary row and column operations on a matrix $A \in \mathbb{C}^{m \times n}(n)$. Similar to their counterparts for real matrices, elementary row (column) operations on A are limited to:

1. interchanging two rows (columns)
2. adding a polynomial multiple of a row (column) to another row (column)
3. multiplying a row (column) by a complex scalar not equal to zero.

As with real matrices, an elementary row (column) operation on $A \in \mathbb{C}^{m \times n}(\mathbb{C})$ is equivalent to left (right) multiplication of A by the corresponding elementary row (column) matrix. For example, interchanging the rows of (3.2) can be accomplished by multiplying A on the left by

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

obtained by interchanging the rows of $I_{2 \times 2}$. Another example: multiplying the first column of (3.2) by x^2 and adding it to the second column can be accomplished by multiplying A on the right by

$$\begin{bmatrix} 1 & x^2 \\ 0 & 1 \end{bmatrix}$$

obtained by multiplying the first column of $I_{2 \times 2}$ by x^2 and adding it to the second column. Again, as with real matrices, the determinant of an elementary matrix over $\mathbb{C}(\mathbb{C})$ is a nonzero complex number, and thus the elementary matrix always has an inverse. It is for this reason that the third elementary operation is limited to multiplication by a scalar for if polynomial multiples were allowed, the determinant of the resulting elementary matrix may be a polynomial with degree ≥ 1 . The matrix would then fail to have an inverse at the roots of its determinant. A square matrix whose determinant equals ± 1 is called unimodular, and so is invertible.

Both elementary row and column operations are used on A to find the general solution to (3.1). Though two different

cases must be considered--A does and doesn't have constant rank--both follow the same approach: find unimodular matrices P and Q that reduce A to a form yielding the general solution of (3.1). The two cases will be analyzed separately.

$AX = b$, rank(A) constant. The aim here is to transform $A \in \mathbb{C}^{m \times n}$, using elementary operations, into the $m \times n$ matrix

$$A_1 = \begin{bmatrix} I_{p \times p} & 0_{p \times n-p} \\ 0_{m-p, p} & 0_{m-p, n-p} \end{bmatrix} \quad (3.3)$$

where p is the rank of A. The main question thus becomes, "What are the unimodular matrices P and Q such that $PAQ = A_1$?" Once P and Q are found, the general solution of (3.1) will follow, as this chapter will show. That such P and Q do exist was recently proven by E. Sontag [31]:

THEOREM 3.0 (Sontag): The following statements are equivalent for a matrix $A = A(n)$ over $R = \mathbb{C}(n)$:

- a. There exists a matrix B over R such that $ABA = A$ and $BAB = B$. B is called the weak generalized inverse (or the $\{1,2\}$ -generalized inverse) of A.
- b. There exists square unimodular matrices P and Q over R such that $A_1 = PAQ$.
- c. As a function of the the complex variables $n = (n_1, \dots, n_n)$, rank(A) is constant.

Proof: See [31]. ■

A method for determining P and Q has been developed by Dr. John Jones Jr., AFIT, and works by keeping a 'cumulative' record of all the elementary row and column operations performed in reducing A to A_1 . The "Jones ST

Method" begins by forming the matrix

$$A_2 = \left[\begin{array}{c|c} A_{m \times n} & I_{m \times m} \\ \hline I_{n \times n} & O_{n \times m} \end{array} \right] \quad (3.4)$$

Subsequent elementary operations are done on A_2 until A becomes A_1 . Once this point is reached, the matrix occupying the $I_{m \times m}$ block is P , and the matrix occupying the $I_{n \times n}$ block is Q . That these two blocks do indeed contain P and Q is easily seen: P and Q will be the respective products of the elementary row and column transformations done in reducing A to A_1 ; since $I_{m \times m}$ is row adjacent to A , P will be the left multiplier of A , and since $I_{n \times n}$ is column adjacent to A , Q will be A 's right multiplier. Also, both P and Q are invertible, since their determinants are nonzero complex. To see this, consider $P = \pi R_i$ where the R_i 's are the elementary row transformations done in reducing A to A_1 . It follows that

$$\det(P) = \det(\pi R_i) = \pi \det(R_i)$$

Since each $\det(R_i)$ is nonzero complex (the nature of an elementary matrix), so will their product be, and thus $\det(P)$. A similar approach holds for Q .

Example: Reduce to the form of (3.3) the matrix

$$A = \begin{bmatrix} 1 & x & y \\ 0 & 0 & 1 \end{bmatrix} \in \mathbb{C}^{2 \times 3}(x, y)$$

Begin by augmenting A by the identity matrices $I_{2 \times 2}$ and $I_{3 \times 3}$ to form per (3.4)

$$\begin{array}{ccc|cc} 1 & x & y & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ \hline 1 & 0 & 0 & & \\ 0 & 1 & 0 & & \\ 0 & 0 & 1 & & \end{array} \quad \begin{array}{c} \\ \\ 0_{3 \times 2} \end{array}$$

The next step is to get an identity matrix in the upper left hand corner of A, per (3.3). First, interchange the second and third columns to form

$$\begin{array}{ccc|cc} 1 & y & x & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ \hline 1 & 0 & 0 & & \\ 0 & 0 & 1 & & \\ 0 & 1 & 0 & & \end{array} \quad \begin{array}{c} \\ \\ 0_{3 \times 2} \end{array}$$

Next, add $-y$ times the second row and add it to the first to form

$$\begin{array}{ccc|cc} 1 & 0 & x & 1 & -y \\ 0 & 1 & 0 & 0 & 1 \\ \hline 1 & 0 & 0 & & \\ 0 & 0 & 1 & & \\ 0 & 1 & 0 & & \end{array} \quad \begin{array}{c} \\ \\ 0_{3 \times 2} \end{array}$$

Next, add $-x$ times the first column to the third column obtaining

$$\begin{array}{ccc|cc} 1 & 0 & 0 & 1 & -y \\ 0 & 1 & 0 & 0 & 1 \\ \hline 1 & 0 & -x & & \\ 0 & 0 & 1 & & \\ 0 & 1 & 0 & & \end{array} \quad \begin{array}{c} \\ \\ 0_{3 \times 2} \end{array}$$

The process is now complete since A has been reduced to the form of (3.3). The P and Q elementary row and column matrices are read off to be

$$P = \begin{bmatrix} 1 & -y \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad Q = \begin{bmatrix} 1 & 0 & -x \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Thus

$$PAQ = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Since $\det(P) = 1$ and $\det(Q) = -1$, P and Q are unimodular and hence invertible. ■

The above example illustrates a theorem helping to identify the general solution of (3.1) for A of constant rank:

THEOREM 3.1: Let $A \in \mathbb{C}^{m \times n}(p)$ and $\text{rank}(A) = p$ a constant. If P and Q are unimodular and

$$PAQ = \begin{bmatrix} I_{p \times p} & 0_{p \times n-p} \\ 0_{m-p \times p} & 0_{m-p \times n-p} \end{bmatrix}$$

where

$$P = \begin{bmatrix} T \\ M \end{bmatrix} \in \mathbb{C}^{m \times m}(p), \quad Q = [S \ N] \in \mathbb{C}^{n \times n}(p),$$

$$T \in \mathbb{C}^{p \times m}(p), \quad M \in \mathbb{C}^{m-p \times m}(p), \quad S \in \mathbb{C}^{n \times p}(p),$$

$$N \in \mathbb{C}^{n \times n-p}(p)$$

then

$$\begin{bmatrix} A & I_{m \times m} \\ I_{n \times n} & 0_{n \times m} \end{bmatrix} \text{ is similar to } \begin{bmatrix} I_{p \times p} & 0_{p \times n-p} & T_{p \times m} \\ 0_{m-p \times p} & 0_{m-p \times n-p} & M_{m-p \times m} \\ S_{n \times p} & N_{n \times n-p} & 0_{n \times m} \end{bmatrix}$$

$$\begin{aligned} \text{PROOF: } & \begin{bmatrix} P_{m \times m} & 0_{m \times n} \\ 0_{n \times m} & I_{n \times n} \end{bmatrix} \begin{bmatrix} A_{m \times n} & I_{m \times m} \\ I_{n \times n} & 0_{n \times m} \end{bmatrix} \begin{bmatrix} Q_{n \times n} & 0_{n \times m} \\ 0_{m \times n} & I_{m \times m} \end{bmatrix} \\ &= \begin{bmatrix} (PA)_{m \times n} & P_{m \times m} \\ I_{n \times n} & 0_{n \times m} \end{bmatrix} \begin{bmatrix} Q_{n \times n} & 0_{n \times m} \\ 0_{m \times n} & I_{m \times m} \end{bmatrix} \\ &= \left[\begin{array}{c|c} (PAQ)_{m \times n} & P_{m \times m} \\ \hline Q_{n \times n} & 0_{n \times m} \end{array} \right] \quad \begin{array}{l} \text{(from given and)} \\ \text{(substitute for } P \text{ \& } Q \text{)} \end{array} \end{aligned}$$

$$= \left[\begin{array}{cc|c} I_{p \times p} & O_{p \times n-p} & T_{p \times n} \\ O_{m-p \times p} & O_{m-p \times n-p} & M_{m-p \times m} \\ \hline S_{n \times p} & N_{n \times n-p} & O_{n \times m} \end{array} \right]$$

Since P and Q are unimodular,

$$\begin{bmatrix} P_{m \times m} & O_{m \times n} \\ O_{n \times m} & I_{n \times n} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} Q_{n \times n} & O_{n \times m} \\ O_{m \times n} & I_{m \times m} \end{bmatrix}$$

are also unimodular and thus invertible. This is seen by noting that if A and B are square matrices, then

$$\det \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} = \det(A) * \det(B)$$

a consequence of the permutational definition of a determinant (2.13). ■

Example of Theorem 3.1: Let

$$A = \begin{bmatrix} x & y & x^2 + y + 1 \\ x^2y & xy^2 & x^3y + xy^2 + xy \end{bmatrix}$$

Matrices P and Q are sought that reduce A to the form (3.3).

First, augment A by the identity matrices $I_{2 \times 2}$ and $I_{3 \times 3}$ to get

$$\begin{array}{ccc|cc} x & y & x^2 + y + 1 & 1 & 0 \\ x^2y & xy^2 & x^3y + xy^2 + xy & 0 & 1 \\ \hline 1 & 0 & 0 & & \\ 0 & 1 & 0 & & \\ 0 & 0 & 1 & & \end{array} \quad \begin{array}{c} \\ \\ 0_{3 \times 2} \end{array}$$

Next, add $-xy$ times row 1 and add to row 2 to obtain:

$$\begin{array}{ccc|cc} x & y & x^2 + y + 1 & 1 & 0 \\ 0 & 0 & 0 & -xy & 1 \\ \hline 1 & 0 & 0 & & \\ 0 & 1 & 0 & & \\ 0 & 0 & 1 & & \end{array} \quad \begin{array}{c} \\ \\ 0_{3 \times 2} \end{array}$$

Next, add -1 times column 2 to column 3 to get:

$$\begin{array}{ccc|cc}
 x & y & x^2 + 1 & 1 & 0 \\
 0 & 0 & 0 & -xy & 1 \\
 \hline
 1 & 0 & 0 & & \\
 0 & 1 & -1 & & \\
 0 & 0 & 1 & &
 \end{array}
 \quad 0_{3 \times 2}$$

Now add $-x$ times column 1 to column 3 to obtain:

$$\begin{array}{ccc|cc}
 x & y & 1 & 1 & 0 \\
 0 & 0 & 0 & -xy & 1 \\
 \hline
 1 & 0 & -x & & \\
 0 & 1 & -1 & & \\
 0 & 0 & 1 & &
 \end{array}
 \quad 0_{3 \times 2}$$

Switching columns 1 and 3:

$$\begin{array}{ccc|cc}
 1 & y & x & 1 & 0 \\
 0 & 0 & 0 & -xy & 1 \\
 \hline
 -x & 0 & 1 & & \\
 -1 & 1 & 0 & & \\
 1 & 0 & 0 & &
 \end{array}
 \quad 0_{3 \times 2}$$

Adding $-y$ times column 1 to column 2, followed by $-x$ times column 1 to column 3 yields:

$$\begin{array}{ccc|cc}
 1 & 0 & 0 & [1 & 0] >> T \\
 0 & 0 & 0 & [-xy & 1] >> M \\
 \hline
 -x & xy & 1 + x^2 & & \\
 -1 & 1 + y & x & & \\
 1 & -y & -x & & \\
 \wedge & \wedge & \wedge & & \\
 \wedge & & \wedge & & \\
 S & & N & &
 \end{array}
 \quad 0_{3 \times 2}$$

Since A has been reduced to the form (3.3), P and Q have been found, and thus S, T, M, N of Theorem 3.1. It's easy to verify the unimodularity of P and Q . ■

The groundwork is now in place to identify the general form of the solution to (3.1). The following theorem is due to Jones [11]:

THEOREM 3.2: Let $A \in \mathbb{C}^{m \times n}(n)$ of constant rank p and $b \in \mathbb{C}^{m \times v}(n)$. Then $AX = b$ has a solution $X \in \mathbb{C}^{n \times v}(n)$ if and only if $Mb = 0_{m-p \times v}$, M given in Theorem 3.1. The general form of X is given by

$$X = STb + NZ$$

S, T, N as given in Theorem 3.1, and $Z \in \mathbb{C}^{(n-p) \times v}(n)$ arbitrary.

PROOF: Let $P \in \mathbb{C}^{m \times m}(n)$ and $Q \in \mathbb{C}^{n \times n}(n)$ be unimodular matrices such that

$$PAQ = \begin{bmatrix} I_{p \times p} & 0_{p \times n-p} \\ 0_{m-p, p} & 0_{m-p, n-p} \end{bmatrix}$$

as given in Theorem 3.0. Then $AX = b$ has a solution X iff

$PAX = Pb$ has a solution X (by virtue of the unimodularity of P and hence the existence of P^{-1})

iff

$PA(QQ^{-1})X = Pb = (PAQ)(Q^{-1}X)$ has a solution X (since Q is unimodular, there exists Q^{-1})

iff

$(PAQ)Y = Pb$ has a solution $Y = Q^{-1}X$ (3.5)

Let

$$Y = \begin{bmatrix} W_{p \times v} \\ Z_{(n-p) \times v} \end{bmatrix}, \quad P = \begin{bmatrix} T_{p \times m} \\ M_{(m-p) \times m} \end{bmatrix}, \quad Q = \begin{bmatrix} S_{n \times p} & N_{n \times n-p} \end{bmatrix}$$

Then substituting for PAQ , P and Y in (3.5):

$$\begin{bmatrix} I_{p \times p} & 0_{p \times n-p} \\ 0_{m-p, p} & 0_{m-p, n-p} \end{bmatrix} \begin{bmatrix} W \\ Z \end{bmatrix} = \begin{bmatrix} T \\ M \end{bmatrix} b$$

$$\begin{bmatrix} W \\ 0_{m-p \times v} \end{bmatrix} = \begin{bmatrix} Tb \\ Mb \end{bmatrix} \quad (3.6)$$

Since $Y = Q^{-1}X$ from (3.5),

$$X = QY = \begin{bmatrix} S & N \end{bmatrix} \begin{bmatrix} W \\ Z \end{bmatrix} = SW + NZ = STb + NZ$$

Note that (3.6) allows Z to be arbitrary since Z is zeroed out by the second row of PAQ . (3.6) also says that if a

solution exists to $AX = b$, then Mb must equal $0_{m-p \times v}$.
This mandate is referred to as the 'consistency condition'.

■

EXAMPLE (THEOREM 3.2):

Given

$$A = \begin{bmatrix} x & y & x^2 + y + 1 \\ x^2y & xy^2 & x^3y + xy^2 + xy \end{bmatrix}$$

$$b = \begin{bmatrix} x + xy + y + x^2y + y^2 \\ x^2y + xy^2 + x^2y^2 + x^3y^2 + xy^3 \end{bmatrix}$$

solve $AX = b$.

The example for Theorem 3.1 (which reduced A) found that

$$T = [1 \ 0] , M = [-xy \ 1] , S = [-x \ -1 \ 1]^t$$

and

$$N = \begin{bmatrix} xy & 1 + x^2 \\ 1 + y & x \\ -y & -x \end{bmatrix}$$

It is a simple matter to show $Mb = 0$ thereby confirming the existence of a solution for $AX = b$. From Theorem 3.2 it follows that the general solution is

$$X = STb + NZ =$$

$$\begin{bmatrix} -x^2 - x^2y - xy - x^3y - xy^2 \\ -x - xy - y - x^2y - y^2 \\ x + xy + y + x^2y + y^2 \end{bmatrix} + a \begin{bmatrix} xy \\ 1 + y \\ -y \end{bmatrix} + \beta \begin{bmatrix} 1 + x^2 \\ x \\ -x \end{bmatrix}$$

where $a, \beta \in \mathbb{C}(x, y)$ ■

A benefit of Theorem 3.2 is that it easily identifies the basis for the kernel of the linear transformation represented by the $m \times n$ matrix A , a task generally quite tedious. Recall that the kernel of a linear transformation

represented by the matrix A is the set

$$K_A = \{ X : AX = 0_{m \times 1} \}$$

To find the basis of K_A , one must solve the equation

$$AX = 0_{m \times 1}$$

which is a special case of $AX = b$ where $b = 0_{m \times 1}$. From Theorem 3.2, it follows that

$$X = STb + NZ = ST0_{m \times 1} + NZ = NZ$$

Thus the span of the columns of N is K_A . However, since $Q = [S \ N]$ and Q is unimodular, the columns of Q are linearly independent, and thus the columns of N . Since the columns of N is a linearly independent spanning set of K_A , it follows that the columns of N are the basis for the kernel of the linear transformation represented by the matrix A .

$AX = b$, rank(A) not constant. Consider the following matrix $A \in \mathbb{Q}^{3 \times 3}(x, y, z)$ which doesn't have constant rank:

$$A = \begin{bmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{bmatrix} \quad (3.7)$$

$\text{Rank}(A) \in \{0, 1, 2, 3\}$, depending on the values assumed by (x, y, z) : if $x = y = z = 0$, then $\text{rank}(A) = 0$; if $x, y, z \neq 0$, then $\text{rank}(A) = 3$; if two are 0 while the third isn't, $\text{rank}(A) = 1$; if two aren't zero while the third is, then $\text{rank}(A) = 2$. This section addresses the solution of the equation $AX = b$ given $\text{rank}(A)$ not constant.

A result similar to Theorem 3.2 holds for such a matrix A . The approach is practically identical to the

constant rank case: find unimodular matrices P and Q which reduce A to the form

$$PAQ = \begin{bmatrix} I_{p \times p} & O_{p \times n-p} \\ O_{m-p \times p} & \Lambda_{m-p \times n-p} \end{bmatrix}, \quad p \geq 1 \quad (3.8)$$

where p is as large as possible and $\Lambda \in \mathbb{C}^{m-p \times n-p}(p)$. As before, P and Q are found using the Jones ST Method. If such P and Q are found (they may not exist), the following theorem applies:

THEOREM 3.3: Let $A \in \mathbb{C}^{m \times n}(p)$. If P and Q are unimodular and

$$PAQ = \begin{bmatrix} I_{p \times p} & O_{p \times n-p} \\ O_{m-p \times p} & \Lambda_{m-p \times n-p} \end{bmatrix}$$

where

$$P = \begin{bmatrix} T \\ M \end{bmatrix} \in \mathbb{C}^{m \times m}(p), \quad Q = \begin{bmatrix} S & N \end{bmatrix} \in \mathbb{C}^{n \times n}(p),$$

$$T \in \mathbb{C}^{p \times m}(p), \quad M \in \mathbb{C}^{m-p \times m}(p), \quad S \in \mathbb{C}^{n \times p}(p),$$

$$N \in \mathbb{C}^{n \times n-p}(p), \quad \Lambda \in \mathbb{C}^{m-p \times n-p}(p)$$

then

$$\begin{bmatrix} A & I_{m \times m} \\ I_{n \times n} & O_{n \times m} \end{bmatrix} \text{ is similar to } \begin{bmatrix} I_{p \times p} & O_{p \times n-p} & T_{p \times m} \\ O_{m-p \times p} & \Lambda_{m-p \times n-p} & M_{n-p \times m} \\ S_{n \times p} & N_{n \times n-p} & O_{n \times m} \end{bmatrix}$$

PROOF: identical to that of Theorem 3.1. ■

Once P and Q have been found, the following theorem is applied:

THEOREM 3.4: Let $A \in \mathbb{C}^{m \times n}(p)$ and $b \in \mathbb{C}^{m \times 1}(p)$. Then $AX = b$ has a solution $X \in \mathbb{C}^{n \times 1}(p)$ if and only if there exists a $Z \in \mathbb{C}^{n-p \times 1}(p)$ such that $Mb = \Lambda Z$ M, Λ given in Theorem 3.3. In this case, the general form of X is given

by

$$X = STb + NZ$$

for all Z such that $Mb = \Omega Z$ and S, T, N given in Theorem 3.3.

PROOF: Let $P \in \mathbb{C}^{m \times m}(n)$ and $Q \in \mathbb{C}^{n \times n}(n)$ be unimodular matrices such that

$$PAQ = \begin{bmatrix} I_{p \times p} & 0_{p \times n-p} \\ 0_{m-p, p} & \Omega_{m-p, n-p} \end{bmatrix}$$

$AX = b$ has a solution X

iff

$PAX = Pb$ has a solution X (by virtue of the unimodularity of P and hence the existence of A^{-1})

iff

$PA(QQ^{-1})X = Pb = (PAQ)(Q^{-1}X)$ has a solution X (since Q is unimodular, there exists Q^{-1})

iff

$(PAQ)Y = Pb$ has a solution $Y = Q^{-1}X$ (3.9)

Let

$$Y = \begin{bmatrix} W_{p \times v} \\ Z_{n-p \times v} \end{bmatrix}, \quad P = \begin{bmatrix} T_{p \times m} \\ M_{m-p \times m} \end{bmatrix}$$

and

$$Q = [S_{n \times p} \quad N_{n \times n-p}]$$

Then substituting for PAQ , P and Y in (3.9):

$$\begin{bmatrix} I_{p \times p} & 0_{p \times n-p} \\ 0_{m-p, p} & \Omega_{m-p, n-p} \end{bmatrix} \begin{bmatrix} W \\ Z \end{bmatrix} = \begin{bmatrix} T \\ M \end{bmatrix} b$$

$$\begin{bmatrix} W \\ \Omega Z \end{bmatrix} = \begin{bmatrix} Tb \\ Mb \end{bmatrix} \quad (3.10)$$

Since $Y = Q^{-1}X$,

$$X = QY = [S \quad N] \begin{bmatrix} W \\ Z \end{bmatrix} = SW + NZ = STb + NZ$$

subject to

$$\Omega Z = Mb \quad (3.11)$$

per (3.10). This latter mandate is the consistency

condition. That $Z \in \mathbb{C}^{n-p \times v}(n)$ proceeds from (3.9): since

Q has polynomial entries and is unimodular, so too is Q^{-1} . Since X is also required to be in $\Phi^{n \times n}(n)$ the product of Q^{-1} and X must be in $\Phi^{n \times n}(n)$. Since Z is a submatrix of $Y = Q^{-1}X$, $Z \in \Phi^{n-p \times n}(n)$. ■

In general, there is no assurance that the consistency condition (3.11) can be met for a given equation $AX = b$: though A may be reduced to the form (3.8), there may not exist a Z OVER $\Phi(n)$ satisfying the consistency condition $\Omega Z = Mb$. It may be necessary to leave $\Phi(n)$ in order to find a Z satisfying the consistency condition, an act contrary to this paper's aim.

What then can be said about the solvability over $\Phi(n)$ of $AX = b$ when $\text{rank}(A)$ isn't constant? It's possible that A isn't reducible to the form (3.8). For instance the matrix in (3.7): reducing A to the form (3.8) implies that A has a rank no less than p (the first p columns of (3.8) are linearly independent despite the form of Ω). However, the rank of A may be zero if $x = y = z = 0$, and so A isn't reducible to the form (3.8).

This example, however, is symptomatic of a larger problem. Assume that a given matrix A has been reduced as much as possible (no known method exists that insures maximum reducibility of a matrix over $\Phi(n)$, n arbitrary), i.e. the maximum value of p has been found. One is then left with finding a Z satisfying the consistency condition. Ω in turn cannot be reduced, since this would violate the

assumption that A has been reduced as far as is possible. If Ω is square, $\det(\Omega)$ cannot be a nonzero complex number (if it were, $\text{rank}(\Omega) = m - p$ and Ω could be reduced to $I_{m-p} \times 0$, thus giving A constant rank, a contradiction). If $\det(\Omega) \neq 0$ repeated use of Cramer's rule may identify some solutions Z , but these would be local since the roots of $\det(\Omega)$ must be omitted. In general, however, Ω will be rectangular.

Thus finding the general solution of (3.1) given $\text{rank}(A)$ not constant ultimately requires solving (3.11) where Ω cannot be transformed into the form (3.8). The search for Z may be aided by matrices related to Ω , the topic of the next section.

SMITH FORM OF A MATRIX. A matrix $A \in \mathbb{C}^{m \times n}(R)$ is said to be in Smith Form (SF) if and only if

$$A = \left[\begin{array}{ccc|ccc} f_1 & & 0 & & & \\ & \ddots & & & & \\ 0 & & f_j & & 0_{j \times n-j} & \\ \hline & & & & 0_{m-j \times n-j} & \end{array} \right] \quad (3.12)$$

where all off diagonal entries equal zero and f_k is a factor of f_{k+1} . For example

$$A = \begin{bmatrix} x & 0 & 0 \\ 0 & x^2y & 0 \\ 0 & 0 & x^2y^3 \end{bmatrix} \in \mathbb{C}^{3 \times 3}(x, y)$$

is in SF (3.12) because all off diagonal entries are zero, x is a factor of x^2y , and x^2y is a factor of x^2y^3 . Another example is

$$A = \begin{bmatrix} x & 0 \\ 0 & x(x+1) \\ 0 & 0 \end{bmatrix} \in \mathbb{Q}^{3 \times 2}(x, y)$$

which is in SF (3.12) because all off diagonal entries are zero, and x is a factor of $x(x+1)$. The solving of matrix equation (3.1) may be aided if A can be transformed (using unimodular matrices P' and Q') into a matrix having SF. In fact, Smith Forms have already appeared: (3.3) is a special case where $f_k = 1$, $k \leq p$. The main aim here is to use SFs to solve the equation (3.11) $\Omega Z = Mb$ of the last section.

To illustrate, assume that in solving $\Omega Z = Mb$ (derived from some hypothetical original equation $AX = b$), unimodular matrices P' and Q' have been found that give $P'\Omega Q'$ a SF. Thus

$$\begin{aligned} \Omega Z &= Mb \\ P'\Omega Z &= P'Mb \\ P'\Omega(Q'Q'^{-1})Z &= P'Mb \\ (P'\Omega Q')(Q'^{-1}Z) &= P'Mb \end{aligned}$$

Letting

$$\begin{aligned} Y &= Q'^{-1}Z \\ H &= P'Mb \\ F &= P'\Omega Q' \end{aligned} \tag{3.13}$$

the last equality becomes

$$FY = H$$

Since F is in SF, there exists at most one nonzero entry in each row and column of F . Because F and H are known, some entries in Y can be found by dividing the respective F and H factors (if however a given entry of Y isn't compatible with two or more equations, or if the entry doesn't turn out to

be a polynomial, then $\Omega Z = Mb$ has no solution Z .)

Depending on the structure of F , there may be entries of Y which can assume arbitrary values. Y is then substituted into (3.13) to yield $Z = Q'Y$, thus solving (3.11) and in turn (3.1).

Given $A \in \mathbb{C}^{m \times n}(R)$, if there exists unimodular matrices $P' \in \mathbb{C}^{m \times m}(R)$ and $Q' \in \mathbb{C}^{n \times n}(R)$ such that $A' = P'AQ'$ is in SF, then A is said to be equivalent to its Smith Form. It has been proven [1:188] that for every $A \in \mathbb{C}^{m \times n}(x)$, A is equivalent to a unique Smith Form (an algorithm exists [1:192] which finds this). However, such is not always the case for multi-dimensional R . In fact, Lee and Zak [20] prove that a matrix $A \in \mathbb{C}^{m \times n}(x, y)$ is equivalent to its SF iff a certain system of linear polynomial equations has a solution. The conditions under which a matrix in three or more variables is equivalent to its Smith Form remains an open question.

Although the existence of a SF for a given matrix A must be known before a search for it can begin, a method is needed which identifies the unimodular P' and Q' which yield a particular SF of A . As before, the Jones ST Method can be used.

EXAMPLE: Find the Smith Form of the matrix (Frost and Storey [8])

$$A = \begin{bmatrix} s+z & 0 & 1 \\ 0 & s+z & 0 \\ 0 & 0 & s \end{bmatrix} \in \mathbb{C}^{3 \times 3}(s, z)$$

Begin by augmenting A with $I_{3 \times 3}$:

$$\begin{array}{ccc|ccc} s+z & 0 & 1 & 1 & 0 & 0 \\ 0 & s+z & 0 & 0 & 1 & 0 \\ 0 & 0 & s & 0 & 0 & 1 \\ \hline 1 & 0 & 0 & & & \\ 0 & 1 & 0 & & & \\ 0 & 0 & 1 & & & \end{array}$$

Next, add $-(z+s)$ times the third column to the first column to get:

$$\begin{array}{ccc|ccc} 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & s+z & 0 & 0 & 1 & 0 \\ -s(s+z) & 0 & s & 0 & 0 & 1 \\ \hline 1 & 0 & 0 & & & \\ 0 & 1 & 0 & & & \\ -(s+z) & 0 & 1 & & & \end{array}$$

Add $-s$ times row 1 to row 3 obtaining:

$$\begin{array}{ccc|ccc} 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & s+z & 0 & 0 & 1 & 0 \\ -s(s+z) & 0 & 0 & -s & 0 & 1 \\ \hline 1 & 0 & 0 & & & \\ 0 & 1 & 0 & & & \\ -(s+z) & 0 & 1 & & & \end{array}$$

Finally, interchange the first and third columns:

$$\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & s+z & 0 & 0 & 1 & 0 \\ 0 & 0 & -s(s+z) & -s & 0 & 1 \\ \hline 0 & 0 & 1 & & & \\ 0 & 1 & 0 & & & \\ 1 & 0 & -(s+z) & & & \end{array}$$

Since the upper left hand matrix is in Smith Form (i.e., 1 is a factor of $s+z$, in turn a factor of $-s(s+z)$ and off diagonal entries are zero)

$$P' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -s & 0 & 1 \end{bmatrix} \quad Q' = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -(s+z) \end{bmatrix}$$

are the two unimodular (easily verifiable) matrices which
reduce A to its Smith Form. ■

IV Solutions to Lypanov and Riccati Equations

This chapter will consider the solutions to the Lypanov matrix equation

$$AX - XB = C \quad (4.1)$$

and the second order Riccati matrix equation

$$XDX + AX + XB + C = 0 \quad (4.2)$$

where the entries of A, B, C, D are in $\mathbb{C}(n)$. It will be shown that solving these two equations for X hinges upon solving equations of the form $AX = b$ the subject of Chapter III.

Lypanov Equation. (4.1) may be solved using tensor products or similarity transformations. The former approach will be addressed first because it uses the results of Chapter III at the onset. The approach using similarity transformations will then be presented as a lead-in to solving the Riccati equation (4.2), whose solution relies exclusively on these transformations. Whichever method is taken though, the linear equation (3.1) will demand resolution.

Lypanov Equation: Tensor Solution. Multiplication between two matrices is easily extended to that of their tensor product. Given $A = [a_{ij}]$ the tensor product of the matrices A and B , represented by $A \otimes B$, is given by $[a_{ij}B]$. Unlike matrix multiplication, the number of columns in A doesn't have to equal the number of rows in B .

Example: If

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

then

$$A \Delta B = [a_{i,j}B] = \begin{bmatrix} 1B & 2B & 3B \\ 4B & 5B & 6B \end{bmatrix}$$

$$= \left[\begin{array}{cc|cc|cc} 1 & 1 & 2 & 2 & 3 & 3 \\ 1 & 2 & 2 & 4 & 3 & 6 \\ \hline 4 & 4 & 5 & 5 & 6 & 6 \\ 4 & 8 & 5 & 10 & 6 & 12 \end{array} \right] \quad \text{a } 4 \times 6$$

It follows that if A is $m \times n$ and B is $r \times s$, then $A \Delta B$ is $mr \times ns$. ■

The next step in solving (4.1) is to identify the solution of the equation

$$AXB = C \quad (4.3)$$

where $A_{i,r}$ denotes the i 'th row of A and $A_{i,c}$ its i 'th column (the i - j th entry of A is given by $a_{i,j}$ with no comma between subscripts).

THEOREM 4.1: Solving the matrix equation

$$A_{m \times n} X_{n \times p} B_{p \times q} = C_{m \times q}$$

is equivalent to solving the equation

$$Gu = c$$

where

$$G = A \Delta B^T, \quad u = [X_{1,r}, \dots, X_{n,r}]^T, \quad c = [C_{1,r}, \dots, C_{m,r}]^T$$

PROOF: Let $A = [a_{i,j}]$, $B = [b_{i,j}]$,

$$C = [c_{i,j}], \quad X = [x_{i,j}]$$

Since $AXB = C$,

$$\begin{aligned}
C_{1,j} &= A_{1,r} X B_{j,c} \\
&= A_{1,r} [X_{1,r} B_{j,c} \dots X_{n,r} B_{j,c}]^T \\
&= \sum_k [a_{1k} \sum_L (x_{kL} \beta_{Lj})] \\
&= \sum_{k,L} (a_{1k} \beta_{Lj} x_{kL}) \\
&= \sum_L (a_{11} \beta_{Lj} x_{1L}) + \dots + \sum_L (a_{1n} \beta_{Lj} x_{nL}) \\
&= a_{11} \sum_L (\beta_{Lj} x_{1L}) + \dots + a_{1n} \sum_L (\beta_{Lj} x_{nL}) \\
&= [a_{11} [\beta_{1j} \dots \beta_{pj}] \dots a_{1n} [\beta_{1j} \dots \beta_{pj}]] \\
&\quad \quad \quad * [X_{1,r} \dots X_{n,r}]^T \\
&= [a_{11} B_{j,c}^T \dots a_{1n} B_{j,c}^T] * [X_{1,r} \dots X_{n,r}]^T \quad (4.4)
\end{aligned}$$

Since $c = [C_{1,r} \dots C_{m,r}]^T$ is given and from (4.4), the theorem follows. ■

The following theorem is due to P. Lancaster [18]:

Theorem 4.2: Given the matrix equation

$$A_1 X B_1 + A_2 X B_2 = C \quad (4.5)$$

where A_i are $m \times m$, B_i are $n \times n$, X, C are $m \times n$ (with X_i and C_i the rows of X and C respectively), (4.5) is equivalent to solving the equation $Gx = c$ where

$$\begin{aligned}
G &= A_1 \Delta B_1^T + A_2 \Delta B_2^T \\
x &= [X_1 \dots X_m]^T \\
c &= [C_1 \dots C_m]^T
\end{aligned}$$

Proof: Similar to that of Theorem 4.1. ■

The restriction on square A_i and B_i in Theorem 4.2 is easily modified to include rectangular matrices. With this modification, Theorem 4.2 allows for the solution of the Lyapunov equation (4.1) which is a special case of (4.5) with $B_1 = I$ and $A_2 = -I$. Once a given Lyapunov equation has been reduced to an equation of the form $AX = b$ using tensor products, the results of Chapter III can be applied to this latter equation.

Example: Solve the Lyapunov equation

$$AX + XB = C = AXI + IXB \quad 4.3$$

where

$$\begin{aligned}
 A &= \begin{bmatrix} s+z & s \\ z & s-z \end{bmatrix}, \quad B = \begin{bmatrix} -z & -s \\ -z-s & s-z \end{bmatrix} \\
 C &= \begin{bmatrix} sz & s^2 \\ 2s^2-2z^2-sz & 4sz-2z^2-3s^2 \end{bmatrix} \\
 x &= \begin{bmatrix} y & z \\ v & w \end{bmatrix} \quad (4.6)
 \end{aligned}$$

The first step is to reduce the given Lyapunov equation to the form $Gx = c$ so that Theorem 4.2 may be used. Thus

$$G = A \Delta I^T + I \Delta B^T =$$

$$\begin{aligned}
 &\begin{bmatrix} s+z & s \\ z & s-z \end{bmatrix} \Delta \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Delta \begin{bmatrix} -z & -z-s \\ -s & s-z \end{bmatrix} = \\
 &\begin{bmatrix} s+z & 0 & s & 0 \\ 0 & s+z & 0 & s \\ z & 0 & s-z & 0 \\ 0 & z & 0 & s-z \end{bmatrix} + \begin{bmatrix} -z & -z-s & 0 & 0 \\ -s & s-z & 0 & 0 \\ 0 & 0 & -z & -z-s \\ 0 & 0 & -s & s-z \end{bmatrix} = \\
 &\begin{bmatrix} s & -z-s & s & 0 \\ -s & 2s & 0 & s \\ z & 0 & s-2z & -z-s \\ 0 & z & -s & -2s-2z \end{bmatrix}
 \end{aligned}$$

Let $x = [y \ z \ v \ w]^T$ and

$c = [sz \ s^2 \ 2s^2-2z^2-sz \ 4sz-2z^2-3s^2]^T$. Thus the solution

$$x = \begin{bmatrix} y & z \\ v & w \end{bmatrix}$$

to the given Lyapunov equation is the solution to the linear equation

$$\begin{bmatrix} s & -z-s & s & 0 \\ -s & 2s & 0 & s \\ z & 0 & s-2z & -z-s \\ 0 & z & -s & -2s-2z \end{bmatrix} \begin{bmatrix} y \\ z \\ v \\ w \end{bmatrix} = \begin{bmatrix} sz \\ s^2 \\ 2s^2-2z^2-sz \\ 4sz-2z^2-3s^2 \end{bmatrix}$$

Depending on the nature of the G matrix obtained using tensor products, the given Lyapunov equation may or may not be easily solvable. For instance, if G cannot be reduced to the form (3.3), one then tries for form (3.8). Here however, finding the Z's satisfying the consistency condition (3.11) $\Omega Z = Mb$ may be difficult. In any event, the use of tensor products creates a large matrix G, thereby compounding calculations.

The second approach to solving the Lyapunov equation (4.1) uses similarity transformations. Although this method may not explicitly identify the solution to (4.1), it will identify a set of matrices within which the solutions of (4.1) will be found.

Lyapunov Equation: Solution via Similarity

Transformations. The second approach to solving (4.1) proceeds from a few simple observations unrelated to solving a matrix equation. Given an $n \times n$ matrix X and $I = I_{n \times n}$ the matrix

$$E = \begin{bmatrix} I & X \\ 0 & I \end{bmatrix} \quad (4.7)$$

always has $\det(E) = 1$ despite the nature of X (this follows from the permutational definition of a determinant (2.13)).

Since E is unimodular, it has the unimodular inverse

$$E' = \begin{bmatrix} I & -X \\ 0 & I \end{bmatrix} \quad (4.8)$$

which is verified below:

$$EE' = \begin{bmatrix} I & X \\ 0 & I \end{bmatrix} \begin{bmatrix} I & -X \\ 0 & I \end{bmatrix} = \begin{bmatrix} I + XO & -IX + XI \\ 0I + IO & -OX + I^2 \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

($E'E = I_{2n \times 2n}$ likewise follows).

The next observation uses the matrix

$$R = \begin{bmatrix} A & C \\ 0 & B \end{bmatrix} \quad (4.9)$$

where A, B, C are all $n \times n$ matrices:

$$\begin{aligned} ERE^{-1} &= \begin{bmatrix} I & X \\ 0 & I \end{bmatrix} \begin{bmatrix} A & C \\ 0 & B \end{bmatrix} \begin{bmatrix} I & -X \\ 0 & I \end{bmatrix} = \begin{bmatrix} A & C+XB \\ 0 & B \end{bmatrix} \begin{bmatrix} I & -X \\ 0 & I \end{bmatrix} \\ &= \begin{bmatrix} A & -AX + C + XB \\ 0 & B \end{bmatrix} \end{aligned} \quad (4.10)$$

The above two observations can now be brought together to solve (4.1): the 1-2'th entry of ERE^{-1} above is itself a Lyapunov equation. If it equalled zero, then X would be a solution to the equation

$$-AX + XB + C = 0 \quad \text{or} \quad AX - XB = C$$

which is of the form (4.1). The following theorem due to Jones [11] has now been proven:

Theorem 4.3: If X is a solution to the Lyapunov equation

$$AX - XB = C$$

where $A, B, C, X \in \mathbb{C}^{n \times n}(\mathbb{C})$ then

$$R = \begin{bmatrix} A & C \\ 0 & B \end{bmatrix} \text{ is similar to } R' = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \quad (4.11)$$

Proof: previous discussion. ■

The following more powerful theorem was proven by Roth [30] for matrices whose entries are complex numbers:

Theorem 4.4: If $A, B, C \in \mathbb{C}^{n \times n}$, then $AX - XB = C$ has a solution $X \in \mathbb{C}^{n \times n}$ if and only if R and R' in (4.11) are similar.

Proof: See [30]. ■

Although Theorem 4.3 is the core result leading to a solution of (4.1), the final assault upon (4.1) will use polynomials of a matrix. For this reason, the following theorems are presented.

Theorem 4.5: Let A, B be $n \times n$ matrices such that B^{-1} exists. Then for $n \in \mathbb{Z}^+$

$$(BAB^{-1})^n = BA^nB^{-1} \quad (4.12)$$

Proof: Inductive reasoning will be used. (4.12) is trivially true for $n = 1$. For $n = 2$,

$$\begin{aligned} (BAB^{-1})^2 &= (BAB^{-1})(BAB^{-1}) = BA(B^{-1}B)AB^{-1} \\ &= BA(I)AB^{-1} = BA^2B^{-1} \end{aligned}$$

and (4.12) is again true. Assume (4.12) is true for $n = k$

Thus

$$(BAB^{-1})^k = BA^kB^{-1}$$

and so

$$\begin{aligned} (BAB^{-1})^{k+1} &= (BAB^{-1})^k(BAB^{-1}) = BA^kB^{-1}(BAB^{-1}) \\ &= BA^k(B^{-1}B)AB^{-1} = BA^{k+1}B^{-1} \end{aligned}$$

and so (4.12) holds true for $k + 1$ and therefore for all

positive integers. ■

Theorem 4.6: Let $f(x)$ be a polynomial in the variable x . Let A, B be $n \times n$ matrices such that B^{-1} exists. Then

$$f(BAB^{-1}) = B f(A) B^{-1}$$

Proof: Let $f(x) = a_0 + \sum_{k=1, n} a_k x^k$ Then

$$\begin{aligned} f(BAB^{-1}) &= a_0 I + \sum_{k=1, n} a_k (BAB^{-1})^k = (\text{from Theorem 4.5}) \\ &= a_0 B B^{-1} + \sum_{k=1, n} a_k B A^k B^{-1} \\ &= B a_0 B^{-1} + B (\sum_{k=1, n} a_k A^k) B^{-1} \\ &= B (a_0 I + \sum_{k=1, n} a_k A^k) B^{-1} \\ &= B f(A) B^{-1} \quad \blacksquare \end{aligned}$$

Theorem 4.7: Let

$$R = \begin{bmatrix} A & 0 \\ B & C \end{bmatrix}$$

A, B, C $n \times n$ matrices. Then for $K \in \mathbb{Z}^+$

$$R^K = \begin{bmatrix} A^K & 0 \\ * & C^K \end{bmatrix} \quad (4.13)$$

* an expression in A, B, C .

Proof: Inductive reasoning will be used. (4.13) is trivially true for $i = 1$, where $* = B = 0A + 1B + 0C$

For $i = 2$,

$$R^2 = \begin{bmatrix} A & 0 \\ B & C \end{bmatrix} \begin{bmatrix} A & 0 \\ B & C \end{bmatrix} = \begin{bmatrix} A^2 & 0 \\ BA+CB & C^2 \end{bmatrix}$$

and (4.13) holds true again. Let (4.13) be true for $n = k$

Then

$$R^{k+1} = R^k R = \begin{bmatrix} A^k & 0 \\ * & C^k \end{bmatrix} \begin{bmatrix} A & 0 \\ B & C \end{bmatrix} = \begin{bmatrix} A^{k+1} & 0 \\ * & C^{k+1} \end{bmatrix}$$

and thus (4.13) is true for all positive integers. ■

Theorem 4.8: Let

$$R = \begin{bmatrix} A & 0 \\ B & C \end{bmatrix}$$

A, B, C $n \times n$ matrices. If $f(x)$ is a polynomial in x , then

$$f(R) = \begin{bmatrix} f(A) & 0 \\ * & f(C) \end{bmatrix} \quad (4.14)$$

* an expression in A, B, C

Proof: Inductive reasoning will be used. Let

$$f(x) = a_0 + \sum_{k=1}^n a_k x^k \quad \text{For } n = 1, f(R) = a_0 + a_1 R =$$

$$a_0 \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} + a_1 \begin{bmatrix} A & 0 \\ B & C \end{bmatrix} = \begin{bmatrix} a_0 I + a_1 A & 0 \\ a_1 B & a_0 I + a_1 C \end{bmatrix} \\ = \begin{bmatrix} f(A) & 0 \\ * & f(C) \end{bmatrix}$$

where

$$* = a_1 B = a_1 B + 0A + 0C$$

and so (4.14) holds.

$$\text{For } n = 2, f(R) = a_0 I + a_1 R + a_2 R^2$$

$$= a_0 \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} + a_1 \begin{bmatrix} A & 0 \\ B & C \end{bmatrix} + a_2 \begin{bmatrix} A^2 & 0 \\ * & C^2 \end{bmatrix} \quad (\text{from Theorem 4.7}) \\ = \begin{bmatrix} a_0 I + a_1 A + a_2 A^2 & 0 \\ * & a_0 I + a_1 C + a_2 C^2 \end{bmatrix} \\ = \begin{bmatrix} f(A) & 0 \\ * & f(C) \end{bmatrix}$$

and (4.14) remains true. Let (4.14) be true for $n = v$

$$\text{If } f_1(x) = [a_0 + \sum_{k=1}^v a_k x^k] \quad \text{and}$$

$$f(x) = f_1(x) + a_{v+1} x^{v+1} \quad \text{then}$$

$$f(R) = \begin{bmatrix} f_1(A) & 0 \\ * & f_1(C) \end{bmatrix} + a_{v+1} R^{v+1} \\ = \begin{bmatrix} f_1(A) & 0 \\ * & f_1(C) \end{bmatrix} + a_{v+1} \begin{bmatrix} A^{v+1} & 0 \\ * & C^{v+1} \end{bmatrix}$$

$$= \begin{bmatrix} f_1(A) + a_{v+1}A^{v+1} & 0 \\ * & f_1(C) + a_{v+1}C^{v+1} \end{bmatrix}$$

$$= \begin{bmatrix} f(A) & 0 \\ * & f(C) \end{bmatrix}$$

and so (4.14) is true for all positive integers. ■

One more theorem remains to be cited before the assault on (4.1) can begin. The Hamilton-Cayley Theorem will permit the identification of a pair of linear equations whose common solutions will contain those of (4.1).

Theorem 4.9 (Hamilton-Cayley): Given A an $n \times n$ matrix and $\xi(\nu) = \det(A - \nu I_{n \times n})$ the characteristic equation of A, then

$$\xi(A) = 0_{n \times n}$$

Proof: see Nering [26:100] ■

The tools are now in place with which to solve (4.1). The following theorem is due to Jones [11]:

Theorem 4.10: Given the Lyapunov equation

$$AX - XB = C \quad A, B, C \in \mathbb{C}^{n \times n} \quad (4.15)$$

let R and R' be defined as in (4.11) and f_A the characteristic equation of A. If

$$f_A(R) = \begin{bmatrix} U & M \\ V & N \end{bmatrix} \quad (4.16)$$

then

$$\begin{aligned} U + XV &= 0 \\ M + XN &= 0 \end{aligned} \quad (4.17)$$

Thus a solution to (4.15) will be found among the common solutions of the pair of equations (4.17).

Proof: From the discussion preceeding Theorem 4.3, it was seen that

$$ERE^{-1} = \begin{bmatrix} I & X \\ 0 & I \end{bmatrix} \begin{bmatrix} A & C \\ 0 & B \end{bmatrix} \begin{bmatrix} I & -X \\ 0 & I \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} = R'$$

for X satisfying (4.15). By substitution,

$$f_A(ERE^{-1}) = f_A(R') \quad (4.18)$$

From Theorem 4.8,

$$\begin{aligned} f_A(R') &= \begin{bmatrix} f_A(A) & 0 \\ * & f_A(B) \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 \\ * & f_A(B) \end{bmatrix} \quad (\text{Hamilton-Cayley Theorem 4.9}) \end{aligned}$$

From Theorem 4.6

$$\begin{aligned} f_A(ERE^{-1}) &= E f_A(R) E^{-1} = \begin{bmatrix} I & X \\ 0 & I \end{bmatrix} \begin{bmatrix} U & M \\ V & N \end{bmatrix} \begin{bmatrix} I & -X \\ 0 & I \end{bmatrix} \\ &= \begin{bmatrix} U+XV & M+XN \\ V & N \end{bmatrix} \begin{bmatrix} I & -X \\ 0 & I \end{bmatrix} \\ &= \begin{bmatrix} U+XV & -(U+XV)X + (M+XN) \\ V & -VX+N \end{bmatrix} \quad (4.19) \end{aligned}$$

$$= \begin{bmatrix} 0 & 0 \\ * & f_A(B) \end{bmatrix} \quad (\text{from (4.18)}) \quad (4.20)$$

Since two matrices are equal iff respective entries are equal, the 1-1th entries of (4.19) and (4.20) imply

$$U + XV = 0 \quad (4.21)$$

and the 1-2th entries imply

$$\begin{aligned} 0 &= -(U + XV)X + (M + XN) \\ &= 0X + (M + XN) \quad (\text{from (4.21)}) \\ &= M + XN \end{aligned}$$

Thus a solution of (4.15) will satisfy the pair of equations (4.17). ■

The example used to illustrate the tensor solution to (4.1) will serve to illustrate Theorem 4.10:

Example Theorem 4.10: Solve the Lyapunov equation

$$\begin{aligned} AX + XB &= C \\ &= AX - X(-B) \end{aligned} \quad (4.22)$$

where

$$A = \begin{bmatrix} s+z & s \\ z & s-z \end{bmatrix}, \quad B = \begin{bmatrix} -z & -s \\ -z-s & s-z \end{bmatrix}$$

$$C = \begin{bmatrix} sz & s^2 \\ 2s^2-2z^2-sz & 4sz-2z^2-3s^2 \end{bmatrix}$$

From (4.11),

$$R = \begin{bmatrix} A & C \\ 0 & -B \end{bmatrix} = \begin{bmatrix} s+z & s & sz & s^2 \\ z & s-z & 2s^2-2z^2-sz & 4sz-2z^2-3s^2 \\ 0 & 0 & z & s \\ 0 & 0 & z+s & z-s \end{bmatrix}$$

Next, find $f_A(\mu)$, the characteristic equation of A.

$$\begin{aligned} f_A(\mu) &= \det \begin{bmatrix} s+z-\mu & s \\ z & s-z-\mu \end{bmatrix} \\ &= (s+z-\mu)(s-z-\mu) - sz \\ &= s^2 - sz - s\mu + zs - z^2 - z\mu - \mu s + \mu z + \mu^2 - sz \\ &= s^2 - sz - z^2 + \mu^2 - 2s\mu \end{aligned}$$

Now evaluate $f_A(R)$ (this was done using the program in Appendix A, a BASIC program performing operations on matrices whose entries are multivariate polynomials):

$$\begin{aligned} f_A(R) &= \left[\begin{array}{cc|cc} 0 & 0 & -s^2z + 3s^3 & -5s^3 + 7s^2z - 2sz^2 \\ 0 & 0 & 5sz^2 - 5s^3 - 2z^3 + 2s^2z & 2sz^2 + 8s^3 - 8s^2z \\ \hline 0 & 0 & 2s^2 - 2sz & -3s^2 + 2sz \\ 0 & 0 & -sz - 3s^2 + 2z^2 & 5s^2 - 4sz \end{array} \right] \\ &= \begin{bmatrix} U & M \\ V & N \end{bmatrix} \end{aligned}$$

According to Theorem 4.10, the solution to (4.22) will be found among the common solutions to the pair of equations

$$U + XV = 0 \quad (4.23)$$

$$M + XN = 0 \quad (4.24)$$

Since $U = V = 0$, (4.23) doesn't contribute to the solution.

So the search restricts itself to solutions of the equation

$$M = -XN \quad \text{or}$$

$$\begin{bmatrix} -s^2z + 3s^3 & -5s^3 + 7s^2z - 2sz^2 \\ 5sz^2 - 5s^3 - 2z^3 + 2s^2z & 2sz^2 + 8s^3 - 8s^2z \end{bmatrix} \\ = \begin{bmatrix} w & x \\ y & z \end{bmatrix} \begin{bmatrix} -2s^2 + 2sz & 3s^2 - 2sz \\ sz + 3s^2 - 2z^2 & -5s^2 + 4sz \end{bmatrix}$$

However, in order to apply the techniques of Chapter III, one must have an equation of the form

$$AX = b$$

It is an easy matter to convert (4.24) into the form of (3.1) by observing that $M = -XN$ implies $M^T = -N^T X^T$. This latter equation is in the form (3.1), and the techniques of Chapter III can be applied to this transposed equation. ■

It is possible to further restrict the set of equations within which the solution of (4.15) belongs:

Theorem 4.11: Given the Lyapunov equation

$$AX - XB = C \quad A, B, C \in \Phi^{n \times n}(\mathbb{R})$$

let R and R' be defined as in (4.11) and f_B the characteristic equation of B . If

$$f_B(R) = \begin{bmatrix} U & M \\ V & N \end{bmatrix}$$

then

$$\begin{aligned} N - VX &= 0 \\ M - UX &= 0 \end{aligned} \quad (4.25)$$

Thus a solution to (4.15) will be found among the common

solutions of the pair of equations (4.25).

Proof: The proof is similar to that of Theorem 4.10.

From the discussion preceeding Theorem 4.3, it was seen that

$$ERE^{-1} = \begin{bmatrix} I & X \\ 0 & I \end{bmatrix} \begin{bmatrix} A & C \\ 0 & B \end{bmatrix} \begin{bmatrix} I & -X \\ 0 & I \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} = R'$$

for X satisfying (4.15). By substitution,

$$f_B(ERE^{-1}) = f_B(R') \quad (4.26)$$

From Theorem 4.8,

$$\begin{aligned} f_B(R') &= \begin{bmatrix} f_B(A) & 0 \\ * & f_B(B) \end{bmatrix} \\ &= \begin{bmatrix} f_B(A) & 0 \\ * & 0 \end{bmatrix} \quad (\text{Hamilton-Cayley Theorem 4.9}) \quad (4.27) \end{aligned}$$

From Theorem 4.6

$$\begin{aligned} f_B(ERE^{-1}) &= E f_B(R) E^{-1} = \begin{bmatrix} I & X \\ 0 & I \end{bmatrix} \begin{bmatrix} U & M \\ V & N \end{bmatrix} \begin{bmatrix} I & -X \\ 0 & I \end{bmatrix} \\ &= \begin{bmatrix} U+XV & M+XN \\ V & N \end{bmatrix} \begin{bmatrix} I & -X \\ 0 & I \end{bmatrix} \\ &= \begin{bmatrix} U+XV & -(U+XV)X + (M+XN) \\ V & -VX+N \end{bmatrix} \\ &= \begin{bmatrix} U+XV & X(N-VX) + M - UX \\ V & N - VX \end{bmatrix} \quad (\text{rearrange 1-2th entry}) \quad (4.28) \\ &= \begin{bmatrix} f_B(A) & 0 \\ * & 0 \end{bmatrix} \quad (\text{from (4.27)}) \quad (4.29) \end{aligned}$$

the 2-2th entries of (4.28) and (4.29) imply

$$N - VX = 0 \quad (4.30)$$

and for the 1-2th entries,

$$\begin{aligned} 0 &= X(N-VX) + M - UX \\ &= 0X + (M - UX) \quad (\text{from (4.30)}) \\ &= M - UX \end{aligned}$$

Thus a solution of (4.15) will satisfy the pair of equations (4.25). ■

The example for Theorem 4.10 will serve to illustrate Theorem 4.11:

Example of Theorem 4.11. From (4.22) begin by finding the characteristic equation for

$$-B = \begin{bmatrix} z & s \\ z+s & z-s \end{bmatrix}$$

$$\begin{aligned} f_{-B}(\mu) &= \det(-B - \mu I) = \det \begin{bmatrix} z-\mu & s \\ z+s & z-s-\mu \end{bmatrix} \\ &= (z-\mu)(z-s-\mu) - s(z+s) \\ &= z^2 - zs - z\mu - \mu z + \mu s + \mu^2 - sz - s^2 \\ &= (z^2 - s^2 - 2sz) + (s - 2z)\mu + \mu^2 \end{aligned}$$

Using the program in Appendix A, $f_{-B}(R) =$

$$\begin{bmatrix} s^2 & 3s^2-2sz & -2sz^2+2s^2z+3s^3 & 5s^2z-2s^3-2sz^2 \\ -2z^2+3sz & 4z^2-6sz+s^2 & -5s^2z+2z^3+s^3+sz^2 & -12sz^2+4z^3+10s^2z-s^3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} U & M \\ V & N \end{bmatrix}$$

The pair of equations which need to be satisfied are

$$N - VX = 0 \quad (4.31)$$

$$M - UX = 0 \quad (4.32)$$

(4.31) doesn't contribute to the solution, and so the solution to (4.15) will be found in the solution space of

$$(4.32) \quad M = UX \text{ or}$$

$$\begin{bmatrix} -2sz^2+2s^2z+3s^3 & 5s^2z-2s^3-2sz^2 \\ -5s^2z+2z^3+s^3+sz^2 & -12sz^2+4z^3+10s^2z-s^3 \end{bmatrix}$$

$$= \begin{bmatrix} s^2 & 3s^2-2sz \\ -2z^2+3sz & 4z^2-6sz+s^2 \end{bmatrix} \begin{bmatrix} w & x \\ y & z \end{bmatrix} \quad \blacksquare$$

To summarize, given the Lyapunov equation

$$AX - XB = C \quad A, B, C, X \in \mathbb{C}^{n \times n}(\mathbb{C})$$

and

$$R = \begin{bmatrix} A & C \\ 0 & B \end{bmatrix}$$

and $f_A(\mu)$, $f_B(\mu)$ the characteristic equations of A and B respectively, then X will be a common solution to the pair of equations

$$\begin{aligned} M &= -XN \\ M' &= U'X \end{aligned}$$

where

$$f_A(R) = \begin{bmatrix} U & M \\ V & N \end{bmatrix} \quad \text{and} \quad f_B(R) = \begin{bmatrix} U' & M' \\ V' & N' \end{bmatrix}$$

Riccati Equation. Attention will now be turned towards (4.2), where $A, B, C, D, X \in \mathbb{C}^{n \times n}(n)$. As with the Lyapunov equation, a few insights (not necessarily connected with solving the Riccati equation) will serve to motivate the techniques used to solve (4.2).

If X is an $n \times n$ matrix and $I = I_{n \times n}$, then the matrix

$$E = \begin{bmatrix} X & I \\ I & 0 \end{bmatrix} \tag{4.33}$$

always has a determinant equal to -1 (this follows from the permutational definition of the determinant), and is thus unimodular. E 's unimodular inverse is

$$E' = \begin{bmatrix} 0 & I \\ I & -X \end{bmatrix}$$

$$\text{since } EE' = \begin{bmatrix} X & I \\ I & 0 \end{bmatrix} \begin{bmatrix} 0 & I \\ I & -X \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

and $E'E = I$ likewise follows.

Given the matrix

$$R = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \quad (4.34)$$

where A, B, C, D are $n \times n$ matrices,

$$\begin{aligned} ERE^{-1} &= \begin{bmatrix} X & I \\ I & 0 \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} 0 & I \\ I & -X \end{bmatrix} \\ &= \begin{bmatrix} XA+C & XB+D \\ A & B \end{bmatrix} \begin{bmatrix} 0 & I \\ I & -X \end{bmatrix} \\ &= \begin{bmatrix} XB+D & XA+C-XBX-DX \\ B & A-BX \end{bmatrix} \\ &= \begin{bmatrix} XB+D & -XBX-DX+XA+C \\ B & A-BX \end{bmatrix} \end{aligned} \quad (4.35)$$

It can be seen that the 1-2th entry of ERE^{-1} is in the form of the Riccati equation (4.2). However, to get the 1-2th entry to agree exactly with (4.2) which is

$$\begin{aligned} XDX + AX + XB + C &= 0 \\ &= -XDX - AX - XB - C \end{aligned} \quad (4.36)$$

it will be necessary to modify the R matrix (4.34). The modification is implemented by comparing the form of the 1-2th entry of (4.35)

$$-XBX - DX + XA + C \quad (\text{from } R)$$

with the the second equation in (4.36)

$$-XDX - AX - XB - C \quad (\text{original})$$

By matching the coefficient matrices A, B, C, D for like expressions involving X , the suggested change is from

$$R = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \quad \text{to} \quad R = \begin{bmatrix} -B & D \\ -C & A \end{bmatrix} \quad (4.37)$$

So with this new form of R ,

$$ERE^{-1} = \begin{bmatrix} X & I \\ I & 0 \end{bmatrix} \begin{bmatrix} -B & D \\ -C & A \end{bmatrix} \begin{bmatrix} 0 & I \\ I & -X \end{bmatrix}$$

$$\begin{aligned}
&= \begin{bmatrix} -XB-C & XD+A \\ -B & D \end{bmatrix} \begin{bmatrix} 0 & I \\ I & -X \end{bmatrix} \\
&= \begin{bmatrix} XD+A & -XB-C-XDX-AX \\ D & -B-DX \end{bmatrix} \tag{4.38}
\end{aligned}$$

If X is a solution of the 1-2th entry, then

$$-XB - C - XDX - AX = 0 = XDX + AX + XB + C$$

and X is a solution of (4.2).

The next insight stems from the procedure followed in solving the Lyapunov equation

$$AX - XB = C$$

In Theorems 4.10 and 4.11, a solution to the Lyapunov equation was found among the solutions to the pair of equations (4.17)

$$\begin{aligned}
U + XV &= 0 \\
M + XN &= 0
\end{aligned}$$

(derived from using the characteristic equation of A f_A) and (4.25)

$$\begin{aligned}
N' - V'X &= 0 \\
M' - U'X &= 0
\end{aligned}$$

(derived from using the characteristic equation of B f_B).

Although only one equation from each pair turned out to be significant in solving (4.1), the original pairs point to the solution of the Riccati equation. To obtain (4.17), notice that if

$$\begin{bmatrix} I & X \\ 0 & I \end{bmatrix} \begin{bmatrix} U & M \\ V & N \end{bmatrix} = \begin{bmatrix} U + XV & M + XN \\ V & N \end{bmatrix}$$

is set equal to

$$\begin{bmatrix} 0 & 0 \\ V & N \end{bmatrix} \tag{4.18}$$

then matching the top rows will yield (4.17). The matrix

$$\begin{bmatrix} I & X \\ 0 & I \end{bmatrix}$$

was picked because it helped generate the Lypanov equation in (4.10). As before,

$$f_A(R) = \begin{bmatrix} U & M \\ V & N \end{bmatrix}, \quad R = \begin{bmatrix} A & C \\ 0 & B \end{bmatrix}$$

The development of the pair (4.25) is similar: set

$$\begin{bmatrix} U & M \\ V & N \end{bmatrix} \begin{bmatrix} I & -X \\ 0 & I \end{bmatrix} = \begin{bmatrix} U & -UX + M \\ V & -VX + N \end{bmatrix}$$

equal to

$$\begin{bmatrix} U & 0 \\ V & 0 \end{bmatrix} \quad \text{where} \quad f_B(R) = \begin{bmatrix} U & M \\ V & N \end{bmatrix}$$

to obtain the pair (4.25).

The same approach will now be used to generate solutions to the Riccati equation. Let $f(x)$ be any polynomial in the indeterminate x with degree ≥ 1 . From (4.37)

$$R = \begin{bmatrix} -B & D \\ -C & A \end{bmatrix}$$

$$\text{Let } f(R) = \begin{bmatrix} U & M \\ V & N \end{bmatrix} \quad (4.39)$$

Now the crucial step: as if developing the pair (4.17), let X be such that

$$\begin{aligned} Ef(R) &= \begin{bmatrix} X & I \\ I & 0 \end{bmatrix} \begin{bmatrix} U & M \\ V & N \end{bmatrix} \\ &= \begin{bmatrix} XU+V & XM+N \\ U & M \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ U & M \end{bmatrix} = S \end{aligned} \quad (4.40)$$

That such an X exists is motivated by the previous discussion concerning the Lypanov equation. From (4.40)

$$f(R) = E^{-1}S \text{ or}$$

$$f(R) = \begin{bmatrix} U & M \\ V & N \end{bmatrix} = \begin{bmatrix} 0 & I \\ I & -X \end{bmatrix} \begin{bmatrix} 0 & 0 \\ U & M \end{bmatrix} = \begin{bmatrix} U & M \\ -XU & -XM \end{bmatrix} \quad (4.41)$$

Because $f(x)$ is a polynomial in x , $xf(x) = f(x)x$ and so from (4.41)

$$Rf(R) = \begin{bmatrix} -B & D \\ -C & A \end{bmatrix} \begin{bmatrix} U & M \\ -XU & -XM \end{bmatrix} = \begin{bmatrix} -BU-DXU & -BM-DXM \\ -CU-AXU & -CM-AXM \end{bmatrix} \quad (4.42)$$

and

$$f(R)R = \begin{bmatrix} U & M \\ -XU & -XM \end{bmatrix} \begin{bmatrix} -B & D \\ -C & A \end{bmatrix} = \begin{bmatrix} -UB-MC & UD+MA \\ XUB+XMC & -XUD-XMA \end{bmatrix} \quad (4.43)$$

Since (4.42) and (4.43) are the same matrix

$$BU + DXU = UB + MC \quad (4.44)$$

$$BM + DXM = -UD - MA \quad (4.45)$$

$$CU + AXU = -XUB - XMC \quad (4.46)$$

$$CM + AXM = XUD + XMA \quad (4.47)$$

Within the above four equations lie two solutions to the Riccati equation (4.2). From (4.44) and (4.46),

$$CU + AXU = -X(UB + MC) = -X(BU + DXU) = -XBU - XDXU$$

and so

$$\begin{aligned} CU + AXU + XBU + XDXU &= 0 \\ &= (C + AX + XB + XDX)U \end{aligned} \quad (4.48)$$

If $\det(U)$ is a nonzero complex number, then U^{-1} exists and from (4.48)

$$XDX + AX + XB + C = 0$$

and X satisfies the Riccati equation (4.2). The second solution uses (4.45) and (4.47):

$$CM + AXM = X(UD + MA) = X(-BM - DXM) = -XBM - XDXM$$

and so

$$CM + AXM + XBM + XDXM = 0 = (C + AX + XB + XDX)M$$

If $\det(M)$ is a nonzero complex number, M^{-1} exists and so

$$XDX + AX + XB + C = 0$$

another solution to (4.2).

The following theorem due to Jones has now been proven:

Theorem 4.12: Given $A, B, C, D \in \mathbb{C}^{n \times n}(\mathbb{C})$ and $f(x)$ a polynomial with degree ≥ 1 , let

$$R = \begin{bmatrix} -B & D \\ -C & A \end{bmatrix} \quad \text{and} \quad f(R) = \begin{bmatrix} U & M \\ V & N \end{bmatrix}$$

If $X \in \mathbb{C}^{n \times n}(\mathbb{C})$ satisfies

$$\begin{aligned} XU + V &= 0 \\ XM + N &= 0 \end{aligned} \tag{4.49}$$

and $\det(M) \in \mathbb{C} - \{0\}$, then $XDX + AX + XB + C = 0$. Also,

if $X \in \mathbb{C}^{n \times n}(\mathbb{C})$ satisfies the pair (4.49) and

$\det(U) \in \mathbb{C} - \{0\}$ then $XDX + AX + XB + C = 0$.

Proof: previous discussion. ■

A third solution exists for (4.2) but requires a different setup akin to developing the pair (4.25). Let R be as in (4.37) and $f(R)$ as in (4.39). Let X satisfy

$$\begin{aligned} f(R) E &= \begin{bmatrix} U & M \\ V & N \end{bmatrix} \begin{bmatrix} 0 & I \\ I & -X \end{bmatrix} = \begin{bmatrix} M & U-MX \\ N & V-NX \end{bmatrix} \\ &= \begin{bmatrix} M & 0 \\ N & 0 \end{bmatrix} \end{aligned} \tag{4.50}$$

That such an X exists is motivated by the development of the pair (4.25) for the Lyapunov equation. (4.50) implies

$$f(R) = \begin{bmatrix} U & M \\ V & N \end{bmatrix} = \begin{bmatrix} M & 0 \\ N & 0 \end{bmatrix} \begin{bmatrix} X & I \\ I & 0 \end{bmatrix} = \begin{bmatrix} MX & M \\ NX & N \end{bmatrix} \tag{4.51}$$

Since $Rf(R) = f(R)R$

$$\begin{aligned}
 Rf(R) &= \begin{bmatrix} -B & D \\ -C & A \end{bmatrix} \begin{bmatrix} MX & M \\ NX & N \end{bmatrix} \\
 &= \begin{bmatrix} -BMX + DNX & -BM + DN \\ -CMX + ANX & -CM + AN \end{bmatrix}
 \end{aligned} \tag{4.52}$$

and

$$\begin{aligned}
 f(R)R &= \begin{bmatrix} MX & M \\ NX & N \end{bmatrix} \begin{bmatrix} -B & D \\ -C & A \end{bmatrix} \\
 &= \begin{bmatrix} -MXB - MC & MXD + MA \\ -NXB - NC & NXD + NA \end{bmatrix}
 \end{aligned} \tag{4.53}$$

Equating last rows of (4.52) and (4.53)

$$NXB + NC = CMX - ANX \tag{4.54}$$

$$NXD + NA = -CM + AN \tag{4.55}$$

It follows that

$$\begin{aligned}
 NXB + NC &= (CM - AN)X = (-NXD - NA)X \\
 &= -NXDX - NAX
 \end{aligned}$$

and so

$$NXDX + NAX + NXB + NC = 0 = N(XDX + AX + XB + C)$$

If $\det(N)$ is a nonzero complex number, N^{-1} exists and so

$$XDX + AX + XB + C = 0$$

and a third solution to (4.2) has been found and thus

Theorem 4.13: Given $A, B, C, D \in \mathbb{C}^{n \times n}(\mathbb{C})$ and $f(x)$ a polynomial with degree ≥ 1 , let

$$R = \begin{bmatrix} -B & D \\ -C & A \end{bmatrix} \quad \text{and} \quad f(R) = \begin{bmatrix} U & M \\ V & N \end{bmatrix}$$

If $X \in \mathbb{C}^{n \times n}(\mathbb{C})$ satisfies

$$\begin{aligned}
 U - MX &= 0 \\
 V - NX &= 0
 \end{aligned} \tag{4.56}$$

and $\det(N) \notin \{0\}$, then $XDX + AX + XB + C = 0$.

Proof: previous discussion. ■

The next theorem presents a companion Riccati equation

to that of (4.2) given a particular polynomial $f(x)$:

Theorem 4.14: Let $f(x)$ be a polynomial of degree ≥ 1 and

$$XDX + AX + XB + C = 0 \quad A, B, C, D, X \in \mathbb{C}^{n \times n}(\mathbb{C})$$

with R and $f(R)$ given by (4.37) and (4.39). Then

$$XMX + NX - XU - V = 0$$

Proof: It has already been shown that

$$E = \begin{bmatrix} X & I \\ I & 0 \end{bmatrix} \quad \text{and} \quad E' = \begin{bmatrix} 0 & I \\ I & -X \end{bmatrix}$$

are both unimodular and inverses of one another. From (4.38) and the given,

$$ERE^{-1} = \begin{bmatrix} XD + A & 0 \\ D & -B - DX \end{bmatrix}$$

From Theorem 4.8,

$$f(ERE^{-1}) = \begin{bmatrix} f(XD+A) & 0 \\ * & f(-B-DX) \end{bmatrix} \quad (4.57)$$

$$= Ef(R)E^{-1} \quad (\text{from Theorem 4.6})$$

$$= \begin{bmatrix} X & I \\ I & 0 \end{bmatrix} \begin{bmatrix} U & M \\ V & N \end{bmatrix} \begin{bmatrix} 0 & I \\ I & -X \end{bmatrix}$$

$$= \begin{bmatrix} XU + V & XM + N \\ U & M \end{bmatrix} \begin{bmatrix} 0 & I \\ I & -X \end{bmatrix}$$

$$= \begin{bmatrix} XM + N & XU + V - XMX - NX \\ M & U - MX \end{bmatrix} \quad (4.58)$$

Equating the 1-2th entries of (4.57) and (4.58),

$$XU + V - XMX - NX = 0 = XMX + NX - XU - V \quad \blacksquare$$

V Solutions of the Third Order Riccati Equation $XAXBX + XCX + XD + EX + F = 0$

This chapter will discuss some of the approaches which may yield solutions to the third order Riccati equation

$$XAXBX + XCX + XD + EX + F = 0 \quad (5.1)$$

where $A, B, C, D, E, F, X \in \mathbb{C}^{n \times n}(n)$, n the tuple of variates. The approach for (5.1) will be less general than that for the second order Riccati and Lypanov equations because of the form of (5.1). Difficulties stemming from this form are discussed in the next chapter.

One approach to solving (5.1) is akin to that taken for the second order Riccati. Let A, B, C, D, E, F be known matrices in $\mathbb{C}^{n \times n}(n)$ and

$$R = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

and

$$f(R) = \begin{bmatrix} U & M \\ V & N \end{bmatrix}$$

where $f(x)$ is a given polynomial with degree ≥ 1 . As seen previously, if $X \in \mathbb{C}^{n \times n}(n)$,

$$\begin{bmatrix} I & X \\ 0 & I \end{bmatrix} \begin{bmatrix} U & M \\ V & N \end{bmatrix} \begin{bmatrix} I & -X \\ 0 & I \end{bmatrix} = \begin{bmatrix} U + XV & -(U + XV)X + (XN + M) \\ V & N - VX \end{bmatrix}$$

Suppose $U + XV = 0$ (motivated by its appearance in solving for the Lypanov and second order Riccati equations). Thus

$$U = -XV \quad (5.2)$$

Because $f(R)$ is a polynomial in R , R and $f(R)$ commute. From (5.2),

$$f(R) = \begin{bmatrix} -XV & M \\ V & N \end{bmatrix}$$

It follows that

$$Rf(R) = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} -XV & M \\ V & N \end{bmatrix} = \begin{bmatrix} BV - AXV & AM + BN \\ DV - CXV & CM + DN \end{bmatrix}$$

and

$$f(R)R = \begin{bmatrix} -XV & M \\ V & N \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} MC - XVA & MD - XVB \\ VA + NC & VB + ND \end{bmatrix}$$

Equating the first columns,

$$BV - AXV = MC - XVA \quad (5.3)$$

$$DV - CXV = VA + NC \quad (5.4)$$

VA appears in the right hand sides of (5.3) and (5.4). If MC in (5.3) involved XNC, then a substitution from (5.4) could be made in (5.3). Accordingly, let

$$MC = -XNC - \Sigma \quad (5.5)$$

Then (5.3) becomes

$$\begin{aligned} BV - AXV &= (-XNC - \Sigma) - XVA \\ &= -X(NC + VA) - \Sigma \\ &= -X(DV - CXV) - \Sigma \quad (\text{from (5.4)}) \\ &= -XDV + XCV - \Sigma \end{aligned}$$

or

$$XCV - XDV + AXV - BV - \Sigma = 0 \quad (5.6)$$

Because all terms except Σ end in V, let

$$\Sigma = e_1 V$$

(5.6) then becomes

$$(XCX - XD + AX - B - e_1)V = 0$$

If $\det(V)$ is a complex, non-zero number (easily verified since $f(R)$ is known), then the inverse of V exists and so

$$XCX - XD + AX - B - e_1 = 0 \quad (5.7)$$

Since solutions of the third order Riccati equation are

sought, let $e_1 = -XEXFX$. (5.7) then becomes

$$XEXFX + XCX + AX - XD - B = 0 \quad (5.8)$$

and X is a solution to this third order Riccati equation.

However, since $e_1 = -XEXFX$ and $\Sigma = e_1 V$, it follows that

$\Sigma = -XEXFXV$ and from (5.2), $\Sigma = XEXFU$. From (5.5) it

follows that

$$MC = -XNC - \Sigma = -XNC - XEXFU$$

To summarize, if

$$R = \begin{bmatrix} A & B \\ C & D \end{bmatrix}, \quad f(R) = \begin{bmatrix} U & M \\ V & N \end{bmatrix}$$

$\det(V)$ a non-zero complex number and X satisfies the pair of equations

$$\begin{aligned} U + XV &= 0 \\ -XEXFU - XNC &= MC \end{aligned} \quad (5.9)$$

then X is a solution to the third order Riccati equation

$$XEXFX + XCX + AX - XD - B = 0 \quad (5.10)$$

If $\det(FU)$ is a non-zero complex number, then (5.9) can be reduced to a second order Riccati equation of the form solved in the previous chapter.

Another solution to (5.10) can be gleaned using the above approach. Let $XN + M = 0$. Then

$$Rf(R) = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} U & -XN \\ V & N \end{bmatrix} = \begin{bmatrix} AU + BV & BN - AXN \\ CU + DV & DN - CXN \end{bmatrix} \quad (5.11)$$

and

$$f(R)R = \begin{bmatrix} U & -XN \\ V & N \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} UA - XNC & UB - XND \\ VA + NC & VB + ND \end{bmatrix} \quad (5.12)$$

Equating the second columns of (5.11) and (5.12),

$$-AXN + BN = UB - XND \quad (5.13)$$

$$-CXN + DN = VB + ND \quad (5.14)$$

Noticing that ND appears in the right hand sides, let

$$UB = -XVB + \Sigma \quad (5.15)$$

Then (5.13) becomes

$$\begin{aligned} -AXN + BN &= (-XVB + \Sigma) - XND \\ &= -X(VB + ND) + \Sigma \\ &= -X(DN - CXN) + \Sigma \\ &= -XDN + XCN + \Sigma \end{aligned} \quad (5.16)$$

and so

$$XCN - XDN + AXN - BN + \Sigma = 0 \quad (5.17)$$

Let $\Sigma = e_1 N$. Then (5.17) becomes

$$(XCX - XD + AX - B + e_1)N = 0$$

If $\det(N)$ is a non-zero complex number, then N has an inverse, and so

$$XCX - XD + AX - B + e_1 = 0 \quad (5.18)$$

If $e_1 = XEXFX$ then

$$XCX - XD + AX - B + XEXFX = 0$$

and another solution to the third order Riccati has been found. Since $\Sigma = e_1 N$, $\Sigma = XEXFXN = -XEXFM$. (5.15) then becomes

$$UB = -XVB - XEXFM$$

Thus, if X is a common solution to the pair of equations

$$\begin{aligned} 0 &= XN + M \\ UB &= -XVB - XEXFM \end{aligned}$$

and $\det(N)$ is non-zero complex, then

$$XEXFX + XCX + AX - XD - B = 0$$

Another approach to solving (5.1) extends the solution via similarity transformation used in the matrix equations presented herein. Though this approach fails to generate

5.4

sets of equations (whose common solutions satisfy (5.1)), it does present relations among the coefficient matrices for a restricted case of (5.1).

Let $A, B, C, D, E, F, G, H \in \mathbb{C}^{n \times n}(\mathbb{R})$. Then given $X \in \mathbb{C}^{n \times n}(\mathbb{R})$,

$$\begin{aligned} & \begin{bmatrix} I & X \\ 0 & I \end{bmatrix} \begin{bmatrix} E & F \\ G & H \end{bmatrix} \begin{bmatrix} I & X \\ 0 & I \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} I & -X \\ 0 & I \end{bmatrix} \\ = & \begin{bmatrix} E+XG & F+XH \\ G & H \end{bmatrix} \begin{bmatrix} A+XC & -AX-XCX+B+XD \\ C & D-CX \end{bmatrix} \\ = & \mathfrak{L} \end{aligned}$$

By multiplying these matrices, the following entries of \mathfrak{L} are obtained:

$$1-1: EA + EXC + XGA + XGXC + FC + XHC \quad (5.19)$$

$$1-2: -XGXCX - X(GA+HC)X - (EA+FC)X + X(GB+HD) + (EB+FD) + (EXD-EXCX+XGXD) \quad (5.20)$$

$$1-3: GA + GXC + HC \quad (5.21)$$

$$1-4: GB + GXD - GAX - GXCX + HD - HCX$$

Of the entries, the only one in which the "cubic" of the third order Riccati appears is (5.20), that is, $XGXCX$. If it weren't for the last term in parenthesis in (5.20), the form of (5.20) would be the same as that of (5.1). In order to eliminate this last term and preserve the structural integrity of (5.20), let $D = E = 0$. It follows that

$$\mathfrak{L} = \begin{bmatrix} XGA+XGXC+FC+XHC & -XGXCX-X(GA+HC)X-FCX+XGB \\ GA+GXC+HC & GB-GAX-GXCX-HCX \end{bmatrix} \quad (5.22)$$

If X were a solution to $\mathfrak{L}(1,2)$, then X would be a solution to a third order Riccati equation of the form

$$XAXBX + XCX + XD + EX = 0 \quad (5.23)$$

which is a special case of (5.1). From the remarks preceeding (5.19), it follows that (since $D = E = 0$)

$$\tilde{z} = \begin{bmatrix} I & X \\ 0 & I \end{bmatrix} \begin{bmatrix} 0 & F \\ G & H \end{bmatrix} \begin{bmatrix} I & X \\ 0 & I \end{bmatrix} \begin{bmatrix} A & B \\ C & 0 \end{bmatrix} \begin{bmatrix} I & -X \\ 0 & I \end{bmatrix} \quad (5.24)$$

$$= \begin{bmatrix} XGA+XGXC+FC+XHC & 0 \\ GA+GXC+HC & GB-GAX-GXCX-HCX \end{bmatrix} \quad (5.25)$$

Multiplying the three middle matrices in (5.24),

$$\begin{bmatrix} 0 & F \\ G & H \end{bmatrix} \begin{bmatrix} I & X \\ 0 & I \end{bmatrix} \begin{bmatrix} A & B \\ C & 0 \end{bmatrix} = \begin{bmatrix} FC & 0 \\ GA + HC + GXC & GB \end{bmatrix} \quad (5.26)$$

$$= \tilde{z}_1$$

If $f(x)$ is any polynomial of degree ≥ 1 , then from (5.24) and Theorem 4.6,

$$f(\tilde{z}) = \begin{bmatrix} I & X \\ 0 & I \end{bmatrix} f(\tilde{z}_1) \begin{bmatrix} I & -X \\ 0 & I \end{bmatrix}$$

and from Theorem 4.8

$$f(\tilde{z}) = \begin{bmatrix} I & X \\ 0 & I \end{bmatrix} \begin{bmatrix} f(FC) & 0 \\ * & f(GB) \end{bmatrix} \begin{bmatrix} I & -X \\ 0 & I \end{bmatrix} \quad (5.27)$$

$$= \begin{bmatrix} f(XGA+XGXC+FC+XHC) & 0 \\ * & f(GB-GAX-GXCX-HCX) \end{bmatrix}$$

Since $f(\tilde{z})$ and $f(\tilde{z}_1)$ are similar matrices, they have the same determinants. From the permutational definition of the determinant (2.13),

$$\det(f(\tilde{z})) = \det[f(XGA+XGXC+FC+XHC)] * \det[f(GB-GAX-GXCX-HCX)] \quad (5.28)$$

and

$$\det(f(\tilde{z}_1)) = \det[f(FC)] * \det[f(GB)] \quad (5.29)$$

and thus

$$\det[f(FC)] * \det[f(GB)] = \det[f(XGA+XGXC+FC+XHC)] * \det[f(GB-GAX-GXCX-HCX)]$$

The following theorem has now been proven:

Theorem 5.1: Given $A, B, C, F, G, H \in \Phi^{n \times n}(n)$ and

$X \in \Phi^{n \times n}(n)$ a solution to

$$-XGX CX - X(GA + HC)X - FCX + XGB = 0$$

then

$$\det[f(FC)] * \det[f(GB)] = \frac{\det[f(XGA + XGX CX + FC + XHC)]}{\det[f(GB - GAX - GX CX - HCX)]}$$

Proof: previous discussion. ■

Theorem 5.1 may be extended to a Riccati equation of the form

$$XA_1XA_2X + XA_3X + A_4X + XA_5 = 0 \quad (5.30)$$

To get this equation in the form (of the given in Theorem 5.1)

$$-XGX CX - X(GA + HC)X - FCX + XGB = 0$$

equating coefficient matrices is done. Thus

$$A_1 = -G \quad (5.31)$$

$$A_2 = C \quad (5.32)$$

$$A_3 = -(GA + HC) = A_1A - HA_2 \quad (5.33)$$

$$A_4 = -FC = -FA_2 \quad (5.34)$$

$$A_5 = GB = -A_1B \quad (5.35)$$

Since G and C are readily known, it remains to determine A, H, F and B. From the above equations, one may have considerable latitude in choosing A, H, F, B. From (5.35), B satisfies the equation $A_1B = -A_5$. Likewise, F satisfies the equation $FA_2 = -A_4$. Lastly, A and H satisfy the Lyapunov equation (5.33)

$$A_3 = A_1A - HA_2$$

By proper choice of A, B, F, H, the challenge of finding solutions to (5.30) via Theorem 5.1 may be made easier.

VI Conclusions and Recommendations

This paper sought the solution X to four types of matrix equations: the linear equation

$$AX = b$$

the Lypanov equation

$$AX - XB = C$$

the second-order Riccati equation

$$XDX + AX + XB + C = 0$$

and the third-order Riccati equation

$$XAXBX + XCX + XD + EX + F = 0$$

where the entries of all matrices are restricted to being multivariate polynomials over the complex numbers. The method of solution involved two phases:

1. identify a similarity transformation on a matrix Z in which is embedded the coefficient matrices of the above equations. The transformation gives a matrix whose entries include the equation being solved.
2. identify a polynomial which, when its argument is the matrix Z in (1), gives a matrix whose entries yield pairs of linear equations. The common solutions of these pairs of equations will contain those of the given matrix equation. The choice of polynomial may yield different solutions.

Since the search for solutions ends in solving a linear equation, a method was presented which may identify the general solution of the given linear equation. The method's success depends on if the matrix A in $AX = b$ is reducible to a Smith Form. The similarity transformation of (1) is given by $E Z E^{-1}$ where E equals

$$\begin{bmatrix} I & X \\ 0 & I \end{bmatrix}$$

(which is unimodular) and Σ for the Lyapunov equation equals

$$\begin{bmatrix} A & C \\ 0 & B \end{bmatrix}$$

and Σ for the second order Riccati equals

$$\begin{bmatrix} -B & D \\ -C & A \end{bmatrix}$$

The third order Riccati presented difficulties because its form wasn't adaptable to the method of similarity transformations. Nevertheless, a transformation could be done which might yield some solutions. When successful, the solution to the third order Riccati was a common solution to a linear and a second order Riccati equation. The polynomial in (2) can assume any form and when its argument is Σ ,

$$f(\Sigma) = \begin{bmatrix} U & M \\ V & N \end{bmatrix} \quad (6.1)$$

The submatrices U,M,V,N yield pairs of equations which depend on the matrix equation being solved. These pairs in turn yield solutions of the original matrix equation.

Recommendations. The biggest challenge is also the central concern: develop the mathematical theory which will identify whether or not a given matrix, whose entries are multivariate polynomials, is reducible to Smith Form. Though the answer for the two-variate polynomial case is known, the three or more variate case remains an open question. It's felt that this challenge will involve

creative work involving number theory, abstract algebra, and matrix theory. Once the theory identifying the conditions under which reducibility to Smith Form is known, the next hurdle will be to develop a computer algorithm performing the reductions (the Smith Form may not be unique). The papers [8] and [20] give approaches to the two-variate case, and the papers [23] and [33] give insights dealing with the "computer programming algebra" of matrix forms and polynomials. Perhaps there is an underlying linear programming or network formulation to the Smith Form reduction.

In solving the Lypanov and Riccati equations, it was necessary to assume the existence of the inverse to one or more of the submatrices in (6.1). Of course, (6.1) depends on the polynomial $f(x)$ chosen, which in turn dictates the types of solutions found to the original matrix equation. Research needs to be done into the types of polynomials yielding one or more invertible submatrices of (6.1). Perhaps there are equivalence classes of polynomials. Or families of polynomials. Maybe there is no structure. Or perhaps there is a "minimum" polynomial which (in some way) generates all polynomials yielding invertible submatrices. Again, a knowledge of abstract algebra may prove valuable: much literature dealing with polynomial structures is couched in abstract algebraic terms.

Much effort was spent in finding the "proper embedding"

for the third order Riccati equation. Disappointingly limited success was realized. To see what is meant by "proper embedding", notice that in

$$\begin{bmatrix} I & X \\ 0 & I \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} I & -X \\ 0 & I \end{bmatrix} = \begin{bmatrix} A + XC & -AX - XCX + B + XD \\ C & -CX + D \end{bmatrix}$$

a second order Riccati equation is found in the 1-2th position of the matrix on the right. If $C = 0$, then this equation becomes the Lypanov. Thus the Lypanov and second order Riccati equations are "embedded" in the matrix on the right, which in turn comes from the similarity transformation on the left. Since this approach has proven so successful in generating linear equations whose solutions are those to the Lypanov and second order Riccati equations, it is natural to extend the similarity transformation to handle the third order Riccati equation.

Unfortunately, this is more easily said than done. Since the third order Riccati equation

$$XAXBX + XCX + XD + EX + F = 0 \quad (6.2)$$

has six coefficient matrices, the above approach would have to be modified to handle these six. As it stands, it is geared to the four coefficients of the second order Riccati equation. Is it possible to work with matrices which are two sub-matrices deep and wide, or is it necessary to go to higher order matrices, perhaps three submatrices in dimension. If so, what would the unimodular matrix be that effects the similarity transformation? Is the restriction of unimodularity unduly restrictive, mandatory though it

seems? Also, the first term $XAXBX$ of (6.2) presents a challenge: how to introduce the middle X from the transformation $E \Sigma E^{-1}$ while hopefully embedding a third order Riccati equation in the resulting matrix of transition. Perhaps a different type of transformation needs to be done. Instead of a single transform, a string of transforms may be the answer, for example

$$E_1 \Sigma_1 E_2 \Sigma_2 E_3$$

where the Σ_j are matrices whose entries include the coefficient matrices, and the E_j contain the solution X . If this approach is indeed the way to go, how are the E_j 's related. Again, the third order Riccati should be embedded somewhere in the resulting matrix.

Or should it? If not embedded, could other matrix equations which are embedded as a matter of course yield, in some combination, some (or all!) solutions of the third order Riccati? Another concern arises: in the approaches presented herein, a specially selected polynomial $f(x)$ played the crucial role of generating pairs of equations whose common solutions included those of the matrix equation in question. For the third order Riccati, how is the polynomial to be selected? How should the pairs be generated, if indeed this pattern holds? Perhaps triplets of equations arise instead of pairs.

The unsettling thought occurs: maybe similarity transforms are not powerful enough for the third order

Riccati, and whole new approaches are in order. After all, current research into these matrix equations is still in the pioneering stages. Hopefully, the successful resolution to the third order Riccati equation will point the way to solving higher order Riccati equations. However, a spectre appears: in solving for the roots of a polynomial in one variable, the insolvability of the quintic was demonstrated by Abel in the early nineteenth century. Could there be a similar obstacle ahead for matrix equations? In any case, finding solutions to matrix equations will prove challenging and endlessly fascinating--as well as having immediate practical applications.

The last recommendation is a minor one: create a new segment (in the computer program of Appendix A) which will compute the determinant of a given square matrix having multivariate polynomial entries. The method would have to keep calculations down to a minimum, since finding determinants is computationally intensive. One approach, which seems to be the quickest way to find a matrix's determinant, was published by G. Macloskie [22] in 1904. This method could be adapted to that of finding determinants for matrices with polynomial entries.

Appendix A: BASIC program performing matrix operations

The following BASIC program performs operations on rectangular matrices having multivariate polynomial entries over the real numbers. Via menu options, the interactive program allows the user to

- create a matrix
- view a matrix
- transpose a matrix
- add two matrices
- multiply two matrices
- extract the U,M,V,N matrices from a given matrix

The size of matrices which the program can handle is limited only by the amount of memory available on the computer and the parameters in the DIMENSION statement. The program was designed on an IBM PC (DOS 3.0) and was intended to assist the thesis effort.

The program's logic hinges upon the way it recognizes a polynomial in n-variables: as a set of (n+1)-tuples each of whose entries come from the coefficients and exponents of the polynomial. To obtain the tuples, the entire polynomial is re-expressed in "standard form" as a sum of terms, each having all variate and exponent positions appear. For instance, the polynomial in the variates x and y

$$2x^2y - 17xy^3 - 3y + 7 \quad (A.1)$$

is re-expressed in standard form as

$$2x^2y^1 + (-17x^1y^3) + (-3x^0y^1) + (7x^0y^0) \quad (A.2)$$

and is the sum of the four terms $2x^2y^1$, $-17x^1y^3$, $-3x^0y^1$, $7x^0y^0$. Though the order of the variates (i.e., x before y

versus y before x) is arbitrary, the order must be consistent.

Once in standard form, each term is put into its tuple representation

$$(\text{coefficient}, \text{exponent}_1, \dots, \text{exponent}_n) \quad (\text{A.3})$$

with the polynomial represented by the collection of the tuples. For the above example:

TERM	3-tuple
$2x^2y^1$	(2,2,1)
$-17x^1y^5$	(-17,1,5)
$-3x^0y^1$	(-3,0,1)
$7x^0y^0$	(7,0,0)

and so the polynomial

$$2x^2y - 17xy^5 - 3y + 7$$

is represented by the set

$$\{ (2,2,1), (-17,1,5), (-3,0,1), (7,0,0) \}$$

Similarly, the polynomial

$$42xyuvz - 2z^2yx$$

is re-expressed in the standard form (variate order is x,y,z,u,v)

$$42x^1y^1z^1u^1v^1 + (-2x^1y^1z^2u^0v^0)$$

and thus the set of 6-tuples

$$\{ (42,1,1,1,1,1), (-2,1,1,2,0,0) \}$$

Addition and multiplication between two polynomials are easily expressed in terms of their tuple representations provided

- all tuples have the same number of entries
- coordinate positions all correspond to the same variable

Addition. Given two terms in n -variables, let (a, α) and (b, β) be their respective $(n+1)$ -tuple representations where a, b are the coefficients of each term and α, β the exponent vectors. For example, if the terms are $3x^2y$ and $7x^9y^5$, then

$$3x^2y \approx (3, 2, 1), \quad a = 3, \quad \alpha = (2, 1)$$

$$7x^9y^5 \approx (7, 9, 5), \quad b = 7, \quad \beta = (9, 5)$$

Tuple addition is defined by

$$(a, \alpha) \# (b, \beta) = \begin{cases} \{ (a, \alpha), (b, \beta) \} & \alpha \neq \beta \\ (a+b, \alpha) & \alpha = \beta \end{cases} \quad (A.4)$$

The operation $\#$ is clearly commutative. To add two polynomials, the tuples representing the terms are added according to (A.4). For example:

$$\begin{aligned} & (2x^2y - 9xy^2) + (4xy^2 + x^2y) \\ \approx & [(2, 2, 1) \# (-9, 1, 2)] \# [(4, 1, 2) \# (1, 2, 1)] \end{aligned}$$

then, rearranging terms

$$\begin{aligned} &= [(2, 2, 1) \# (1, 2, 1)] \# [(-9, 1, 2) \# (4, 1, 2)] \\ &= (3, 2, 1) \# (-5, 1, 2) \\ &= \{ (3, 2, 1), (-5, 1, 2) \} \\ &\approx 3x^2y - 5xy^2 \quad \blacksquare \end{aligned}$$

Multiplication. Multiplication between two tuples is defined by

$$(a, \alpha) \cdot (b, \beta) = (ab, \alpha + \beta) \quad (A.5)$$

where $\alpha + \beta$ is the familiar operation of addition between two vectors in Euclidean n -space. For example,

$$\begin{aligned}
 (2x^3y^5)(6x^7y^8) &\approx (2,3,5) \cdot (6,7,8) \\
 &= (2 \cdot 6, 3+7, 5+8) \\
 &= (12, 10, 13) \\
 &\approx 12x^{10}y^{13} \quad \blacksquare
 \end{aligned}$$

Using (A.5) the tuple representation of the product between two polynomials is quickly found. Consider the product of the n-term polynomial

$$\text{TERM}_{11} + \dots + \text{TERM}_{1n} \quad (\text{A.6})$$

and the m-term polynomial

$$\text{TERM}_{21} + \dots + \text{TERM}_{2m} \quad (\text{A.7})$$

which is

$$\begin{aligned}
 &\text{TERM}_{11}(\text{TERM}_{21} + \dots + \text{TERM}_{2m}) + \dots \quad (\text{A.8}) \\
 &\quad + \text{TERM}_{1n}(\text{TERM}_{21} + \dots + \text{TERM}_{2m}) \\
 &= (\text{TERM}_{11}\text{TERM}_{21} + \dots + \text{TERM}_{11}\text{TERM}_{2m}) + \dots \\
 &\quad + (\text{TERM}_{1n}\text{TERM}_{21} + \dots + \text{TERM}_{1n}\text{TERM}_{2m})
 \end{aligned}$$

If the value of TERM_{jk} is now changed to the tuple form of the polynomial element TERM_{jk} , then the product of (A.6) and (A.7) expressed in tuple form is

$$\begin{aligned}
 &= (\text{TERM}_{11} \cdot \text{TERM}_{21} \# \dots \# \text{TERM}_{11} \cdot \text{TERM}_{2m}) \# \dots \\
 &\quad \# (\text{TERM}_{1n} \cdot \text{TERM}_{21} \# \dots \# \text{TERM}_{1n} \cdot \text{TERM}_{2m})
 \end{aligned}$$

EXAMPLE:

$$\begin{aligned}
 &(x^2y^3 + 12x^4y^5)(x^6y^7 + 11x^8y^9) \\
 &= (x^2y^3)(x^6y^7) + (x^2y^3)(11x^8y^9) \\
 &\quad + (12x^4y^5)(x^6y^7) + (12x^4y^5)(11x^8y^9) \\
 &\approx [(1,2,3) \cdot (1,6,7)] \# [(1,2,3) \cdot (11,8,9)] \\
 &\quad \# [(12,4,5) \cdot (1,6,7)] \# [(12,4,5) \cdot (11,8,9)] \\
 &= (1,8,10) \# (11,10,12) \\
 &\quad \# (12,10,12) \# (132,12,14) \\
 &= (1,8,10) \# (23,10,12) \# (132,12,14) \\
 &= \{ (1,8,10), (23,10,12), (132,12,14) \} \\
 &\approx x^8y^{10} + 23x^{10}y^{12} + 132x^{12}y^{14} \quad \blacksquare
 \end{aligned}$$

Network. Put in a nutshell, the program keeps track of tuple operations using a system of pointers (program lines 1150-1490). That is, a network tree is constructed which

represents the results of tuple operations. This section assumes knowledge of networks as given in [10:91-124].

The fundamental insight is that a polynomial can be represented by a tree. Specifically, the results of tuple operations is maintained in a network tree whose nodes represent a particular variate in a polynomial term. Nodal "potentials" are the exponent of the corresponding variate. Terminal nodes of the polynomial's tree have a second potential whose value is that of the coefficient of a particular term. The i 'th depth of the tree corresponds to the i 'th variate. For example, the polynomial

$$3x^2yz^{13} - 7x^{12}y^7z + 8x^{12}y^7z^{22}$$

is represented by the tree in Figure 1. All the nodes at depth one are x variates; nodes at depth two are y variates; nodes at depth three are z variates. The numbers in [] are node potentials, and numbers in []* are the coefficient of the given polynomial term. From this example, it can be seen that a polynomial term corresponds to a unique path from node 0 to a given terminal node at depth three, e.g., the term $8x^{12}y^7z^{22}$ has the nodal path 0-4-5-7.

In performing a tuple operation, the order in which the branches are built are term by term. The following example illustrates the sequence in which the program builds branches.

EXAMPLE: Construct the tree representing

$$x^2(1 + y) + y(x^2 + y)$$

$$\begin{aligned} x^2(1 + y) + y(x^2 + y) &= x^2 + x^2y + yx^2 + y^2 \\ &= x^2y^0 + x^2y^1 + y^1x^2 + x^0y^2 \end{aligned} \quad (\text{A.9})$$

Since there are 2 variates x, y the tree will have 2 depths. The first term is x^2y^0 which contributes the first branch 0-1-2 in Figure 2. The second term x^2y^1 contributes the branch 0-1-3 in Figure 3. Note that the second potential of node 3 ([1]*) is 1: the coefficient of the second term x^2y^1 . Because the third term of (A.9) is also x^2y^1 , it doesn't contribute a branch to the tree. However, the second potential at node 3 is increased by 1: the coefficient in the third term x^2y^1 (Figure 4). The fourth term x^0y^2 contributes the branch 0-4-5 in Figure 5. Since there are no more terms in (A.9), the tree in Figure 5 represents the polynomial $x^2(1 + y) + y(x^2 + y)$

Manual input of data. When creating a matrix via the menu prompt to "build a matrix", polynomials are entered term by term in their tuple form with no commas between tuple entries. Each tuple MUST remain on the same line, and each tuple except the last ends with a carriage return; the last tuple ends with a semicolon followed by a carriage return. The program will then prompt for the next polynomial entry and the above protocol is repeated. A sample session inputting the matrix

$$Q = \begin{bmatrix} x^2 + 3xyz + z^2 & 4 + 2x - 3y \\ z^4 + 17y^7 & x^2y^3z^4 \end{bmatrix} \quad (\text{A.10})$$

is given in Figure 6.

Output. The program outputs a polynomial in its tuple

form. Figure 7 is a sample session which printed the matrix $fA(R)$ preceeding (4.23).

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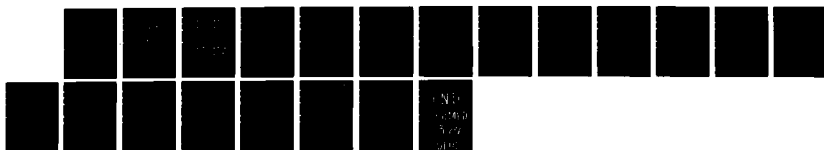
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INST OF TECH WRIGHT-PATTERSON AFB OH SCHOOL OF
ENGINEERING B W COLLETTI DEC 85 AFIT/GOR/MA/85D-1

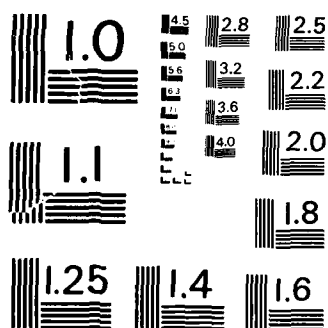
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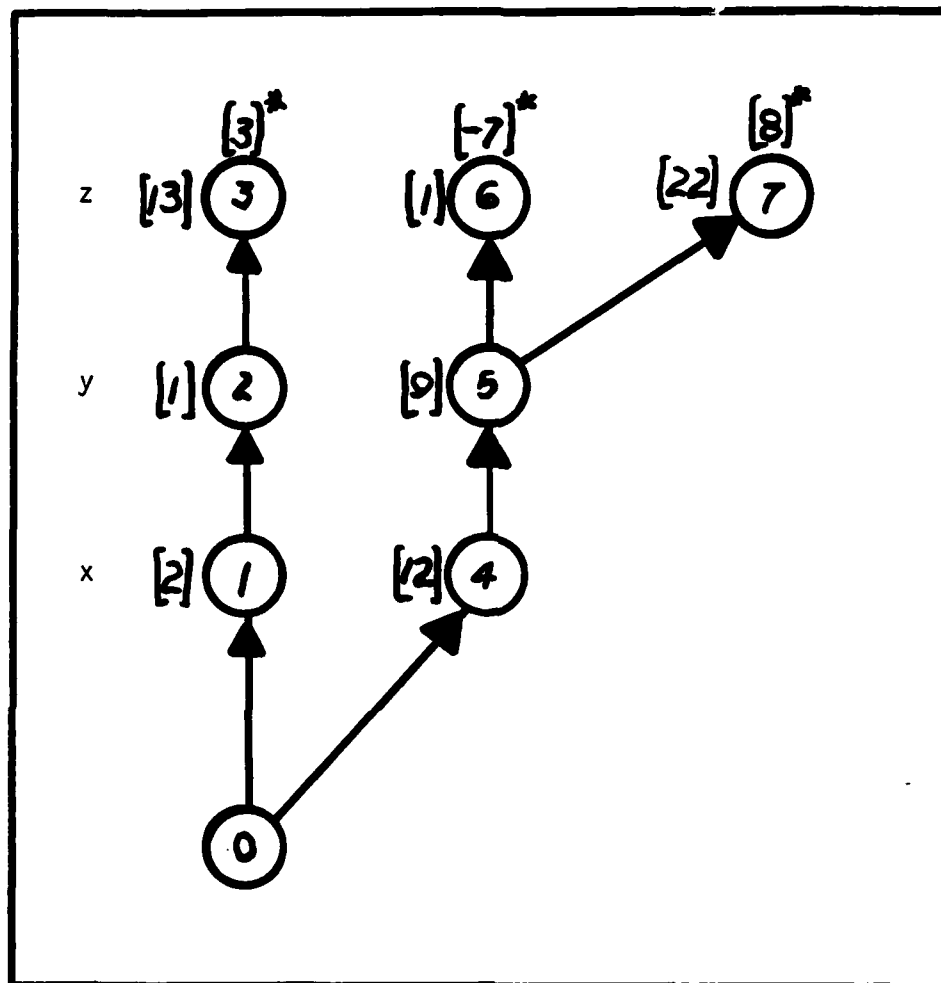


Figure 1. Network tree representing the polynomial

$$3x^2yz^{13} - 7x^{12}y^9z + 8x^{12}y^9z^{22}$$

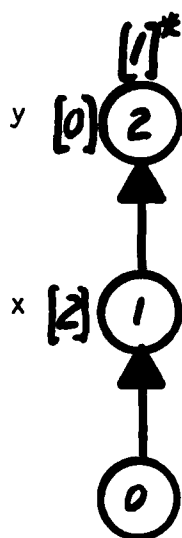


Fig. 2. Branch for x^2

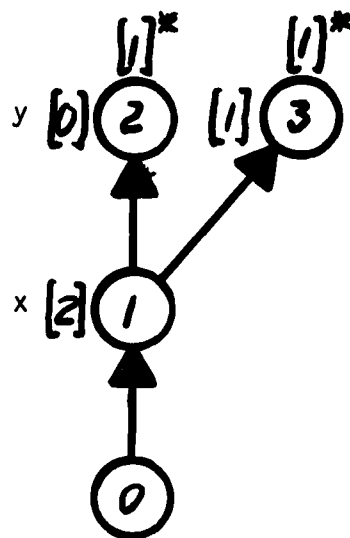


Fig. 3. Branch for $x^2 + x^2y$

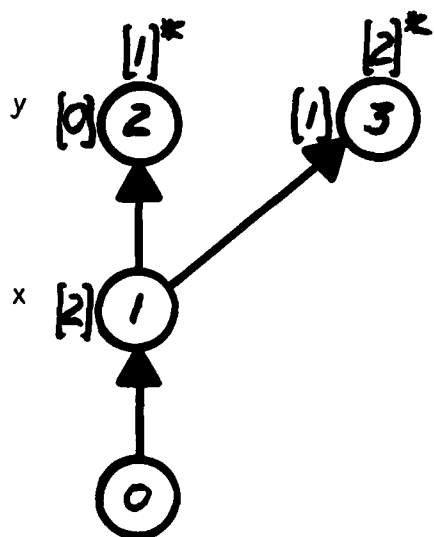


Fig. 4. Branch for $x^2 + 2x^2y$

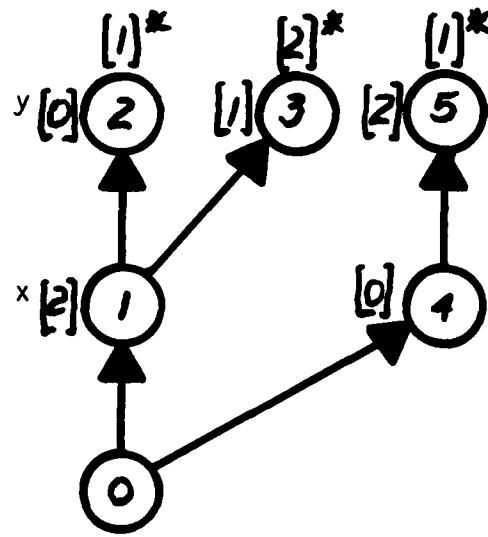


Fig. 5. Branch for $x^2 + 2x^2y + y^2$

```

*****
MENU:
      B  Build a new matrix"
      V  View a matrix"
      .  Transpose a matrix"
      +  Add 2 matrices"
      *  Multiply 2 matrices"
      U  Extract U,M,V,N matrices
*****
? B
Matrix name? B:Q
ROW,COL dimension? 2,2
Number of polynomial variates? 3
Build a square diagonal matrix whose diagonal entries
are equal Y/N? N
*** COEF EXP1 EXP2 ... EXPn ***
      a semicolon ; represents end of polynomial
(1,1):
? 1 2 0 0
? 3 1 1 1
? 1 0 0 2;

(1,2):
? 4 0 0 0
? 2 1 0 0
? -3 0 1 0;

(2,1):
? 1 0 0 6
? 17 0 98 0;

(2,2):
? 1 2 3 4;

```

Figure 6. Session building matrix (A.10)

```

*****
MENU:
    B  Build a new matrix"
    V  View a matrix"
    '  Transpose a matrix"
    +  Add 2 matrices"
    *  Multiply 2 matrices"
    U  Extract U,M,V,N matrices
*****
? V
Screen output Y/N? Y
Matrix to view? B:fA(R)

MATRIX B:fA(R) IS 4 X 4 and has 2 polynomial variates
B:fA(R)( 1, 1):          B:fA(R)( 3, 1):
0 0 0                    0 0 0

B:fA(R)( 1, 2):          B:fA(R)( 3, 2):
0 0 0                    0 0 0

B:fA(R)( 1, 3):          B:fA(R)( 3, 3):
-1 2 1                   2 2 0
 3 3 0                  -2 1 1

B:fA(R)( 1, 4):          B:fA(R)( 3, 4):
-5 3 0                   -3 2 0
 7 2 1                   2 1 1
-2 1 2

B:fA(R)( 2, 1):          B:fA(R)( 4, 1):
0 0 0                    0 0 0

B:fA(R)( 2, 2):          B:fA(R)( 4, 2):
0 0 0                    0 0 0

B:fA(R)( 2, 3):          B:fA(R)( 4, 3):
 5 1 2                   -1 1 1
-5 3 0                   -3 2 0
-2 0 3                   2 0 2
 2 2 1

B:fA(R)( 2, 4):          B:fA(R)( 4, 4):
 2 1 2                   5 2 0
 8 3 0                  -4 1 1
-8 2 1

```

Figure 7. Session viewing the matrix $fA(R)$ preceeding (4.23)

NOTE: the program has been modified to fit within the format requirements of the AFIT Style Guide, and so doesn't represent the correct syntax of the program. However, the adjustments are few and are apparent.

```

10      'CPT Bruce W. Colletti, AFIT/EN; 1 August 1985
20      'The input file for this program (which multiplies 2
30      'matrices having multivariate polynomial entries)
      'has the following
40      'structure:
50      '   line 1:  row, column dimension of matrix A;
      '               number of variates
60      '   line 2:  number of terms in the A(1,1)
      '               polynomial
70      '   line 3:  tuple corresponding to the 1st term of
      '               A(1,1)
80      '   line 4:  tuple corresponding to the 2nd term of
      '               A(1,1)
90      '   line n:  tuple corresponding to the last term
      '               of A(1,1)
100     '   line n+1: number of terms in the A(1,2)
      '               polynomial
110     '   line n+2: tuple corresponding to the 1st term of
      '               A(1,2)
120     '   line n+3: tuple corresponding to the 2nd term of
      '               A(1,2)
130     '   line m:  tuple corresponding to the last term
      '               of A(1,2)
140     '   line m+1: number of terms in the A(1,3)
      '               polynomial
150     '   etc:    matrix entries are entered row at a
      '               time

160     CLS : CLEAR
170     ROWS = 5 : COLS = 5 : TERM = 25 : VAR = 5
180     DIM A(ROWS,COLS,TERM,VAR), B(ROWS,COLS,TERM,VAR),
      B$(20)
190     DIM F(90),POT(90),R(90),BACK(90),M(90),E(VAR)
200     DIM COEFL(ROWS,COLS),COEFR(ROWS,COLS)
210     'A contains the elements of the left matrix and B
      'the elements of
220     'the right (e.g., A + B, A*B):
230     'A(I,J,K,L) the exponent of the L'th variate of the
      'K'th term of the
240     '               I-J'th polynomial entry of A
250     'B(I,J,K,L) the exponent of the L'th variate of the
      'K'th term of the
260     '               I-J'th polynomial entry of B
270     '   BACK(i) the node-back pointer for node i
280     'COEFL(i,j) the number of terms in the polynomial
      'located at
290     '               A(i,j)
300     'COEFR(i,j) the number of terms in the polynomial
      'located at

```

```

310      '      B(i,j)
320      '      E(i)      contains the exponent of the i'th
                        '      variable of the
330      '      product between 2 polynomial terms
340      '      F(i)      forward pointer for node i
350      '      M(i)      the "summed coefficient". This is a
                        '      node in the final
360      '      tree with no forward pointer. It is
                        '      the coefficient of
370      '      that term of the polynomial having the
                        '      path from given
380      '      terminal node to source node
390      '      POT(i)     potential of node i; that is, the
                        '      exponent of the
400      '      variable represented by the node
410      '      R(i)      the right pointer for node i
420      PRINT "*****"
430      PRINT "MENU:"
440      PRINT "      B  Build a new matrix"
450      PRINT "      V  View a matrix"
460      PRINT "      '  Transpose a matrix"
470      PRINT "      +  Add 2 matrices"
480      PRINT "      *  Multiply 2 matrices"
490      PRINT "      U  Extract U,M,V,N matrices"
500      PRINT "*****"
510      INPUT OPT$
520      IF OPT$ = "B" THEN 2410
530      IF OPT$ = "V" THEN 2790
540      IF OPT$ = "U" THEN 3470
550      IF OPT$ = "'" THEN 3130
560      IF ((OPT$ <> "*") AND (OPT$ <> "+")) THEN 420
570      INPUT "Left matrix"; FILEL$
580      INPUT "Right matrix"; FILER$
590      INPUT "Output matrix"; FILEO$
600      IF OPT$ = "+" THEN 620
610      INPUT "α ∈ R in αLEFT*RIGHT"; ALPHA : GOTO 630
620      INPUT "α,β ∈ R in αLEFT + βRIGHT"; ALPHA,BETA
630      FILE$ = FILEL$ : GOSUB 950
640      ROW1 = ROW : COL1 = COL : V1 = V
650      FILE$ = FILER$ : GOSUB 950
660      ROW2 = ROW : COL2 = COL : V2 = V
670      IF V1 <> V2 THEN 1730
680      IF ((OPT$ = "+") AND
        ((ROW1 <> ROW) OR (COL1 <> COL))) THEN 1560
690      IF ((OPT$ = "*") AND (COL1 <> ROW)) THEN 1560
700      OPEN FILEO$ FOR OUTPUT AS #2
710      IF OPT$ = "+" THEN GOSUB 2140
720      PRINT #2, ROW1; COL2; V
730      FOR I1 = 1 TO ROW1
740          FOR I2 = 1 TO COL2
750              PRINT "("; I1; ","; I2; ")"
760              GOSUB 1800
770          FOR I3 = 1 TO COL1

```

```

780          T1 = COEFL(I1,I3) : T2 = COEFR(I3,I2)
790          FOR I4 = 1 TO T1
800              FOR I5 = 1 TO T2
810                  PROD = ALPHA * A(I1,I3,I4,0) *
                        B(I3,I2,I5,0)
820                  FOR I6 = 1 TO V
830                      E(I6) = A(I1,I3,I4,I6) + B(I3,I2,I5,I6)
840                  NEXT I6
850                  GOSUB 1150
860              NEXT I5
870          NEXT I4
880          NEXT I3
890          GOSUB 1870
900          NEXT I2
910      NEXT I1
920      CLOSE #2
930      STOP
940      'Subroutine reads a matrix for +/* routines
950      OPEN FILE$ FOR INPUT AS #1
960      INPUT #1, ROW,COL,V
970      IF ((ROW > ROWS) OR (COL > COLS)) THEN 1620
980      FOR I = 1 TO ROW
990          FOR J = 1 TO COL
1000              INPUT #1, TERMS
1010              IF TERMS > TERM THEN 1680
1020              IF FILE$ = FILEL$ THEN COEFL(I,J) = TERMS
1030              IF FILE$ = FILER$ THEN COEFR(I,J) = TERMS
1040              FOR K = 1 TO TERMS
1050                  FOR L = 0 TO V
1060                      IF FILE$ = FILER$ THEN 1080
1070                      INPUT #1, A(I,J,K,L) : GOTO 1090
1080                      INPUT #1, B(I,J,K,L)
1090                  NEXT L
1100              NEXT K
1110          NEXT J
1120      NEXT I
1130      CLOSE #1
1140      RETURN
1150      'Pointer ordering subroutine
1160      I = 1 : CNODE = F(0) : BK = 0
1170      IF CNODE = 0 THEN 1370
1180      BASE = BK
1190      IF POT(CNODE) = E(I) THEN 1430
1200      BK = CNODE
1210      CNODE = R(CNODE)
1220      IF CNODE <> 0 THEN 1190
1230      NODES = NODES + 1
1240      POT(NODES) = E(I)
1250      R(BK) = NODES
1260      BACK(NODES) = BASE
1270      BK = NODES
1280      FOR J = I+1 TO V
1290          NODES = NODES + 1

```

```

1300      POT(NODES) = E(J)
1310      BACK(NODES) = BK
1320      F(BK) = NODES
1330      BK = NODES
1340      NEXT J
1350      M(BK) = M(BK) + PROD
1360      GOTO 1490
1370      NODES = NODES + 1
1380      F(NODES) = 0
1390      F(BK) = NODES
1400      BACK(NODES) = BK
1410      CNODE = NODES
1420      POT(CNODE) = E(I)
1430      BASE = CNODE
1440      I = I + 1 'I is the same as the Depth of a node
1450      IF (I > V) THEN M(CNODE) = M(CNODE) + PROD :
      GOTO 1490
1460      BK = CNODE
1470      CNODE = F(CNODE)
1480      GOTO 1170
1490      RETURN
1500      'Error routine for too many polynomial variates
1510      PRINT "*****"
1520      PRINT CHR$(7); "DIM statement allows only "; VAR;
      "variables.";
1530      PRINT "  Modify DIM statement";
1540      STOP
1550      'Error routine for ill-dimensioned matrices
1560      PRINT "*****"
1570      PRINT CHR$(7); "Dimensions incompatable"
1580      PRINT "    "; FILEL$ ; " is "; STR$(ROW1); " X ";
      STR$(COL1)
1590      PRINT "    "; FILER$ ; " is "; STR$(ROW); " X ";
      STR$(COL)
1600      STOP
1610      'Error routine for matrices whose dimensions are
      'too large
1620      PRINT "*****"
1630      PRINT CHR$(7); "Modify the dimension statement in
      the program"
1640      PRINT ; "The dimensions of "; FILE$; "is ";
      STR$(ROW);
1650      PRINT " X "; STR$(COL)
1660      STOP
1670      'Error routine for a polynomial having too many
      terms
1680      PRINT "*****"
1690      PRINT CHR$(7); "There are "; TERMS; "terms in
      the ";
1700      PRINT "polynomial, whereas the DIM statement
      allocates "
1710      PRINT TERM; " terms in the polynomial.  Modify
      DIM stmt."

```

```

1720     STOP
1730     PRINT "*****"
1740     PRINT CHR$(7); "Polynomial variates in matrices
        aren't the same"
1750     STOP
1760     'Error for finding U,M,V,N matrices
1770     PRINT CHR$(7); "Source matrix "; FILE1$; " doesn't
        have even";
1780     PRINT " row and column dimensions"
1790     STOP
1800     'Subroutine to clear tree building variables
1810     FOR I = 0 TO NODES
1820         F(I) = 0 : POT(I) = 0 : R(I) = 0
1830         BACK(I) = 0 : M(I) = 0
1840     NEXT I
1850     NODES = 0
1860     RETURN
1870     'Subroutine prints polynomial term
1880     S = 0
1890     FOR I = 1 TO NODES 'find # of terms in polynomial
1900         IF M(I) <> 0 THEN S = S + 1
1910     NEXT I
1920     IF S = 0 THEN 1940
1930     PRINT #2,S : GOTO 2000
1940     PRINT #2,"1";
1950     FOR I = 0 TO V 'handles a zero entry in result
1960         PRINT #2,0;
1970     NEXT I
1980     PRINT #2,""
1990     GOTO 2130
2000     FOR I = 1 TO NODES
2010         IF M(I) = 0 THEN 2120
2020         PRINT #2, M(I); 'coefficient of term
2030         PTR = I
2040         FOR J = V TO 1 STEP -1 'because we are going
                                from the
2050             ZZ(J) = POT(PTR) 'top of the tree down
                                but are
2060             PTR = BACK(PTR) 'writing the exponents
                                in the
2070             NEXT J 'reverse order
2080         FOR J = 1 TO V
2090             PRINT #2, ZZ(J);
2100         NEXT J
2110         PRINT #2,""
2120     NEXT I
2130     RETURN
2140     'Addition subroutine
2150     PRINT #2,ROW1; COL1; V1
2160     FOR I1 = 1 TO ROW1
2170         FOR I2 = 1 TO COL2
2180             PRINT "(", I1; ", "; I2; ")"
2190             GOSUB 1800 'clear tree

```

```

2200 'Send A(I1,I2) up through tree
2210     FOR I4 = 1 TO COEFL(I1,I2)
2220         PROD = ALPHA * A(I1,I2,I4,0)
2230         FOR I6 = 1 TO V
2240             E(I6) = A(I1,I2,I4,I6)
2250         NEXT I6
2260         GOSUB 1150
2270     NEXT I4
2280 'Send B(I1,I2) up through tree
2290     FOR I4 = 1 TO COEFR(I1,I2)
2300         PROD = BETA * B(I1,I2,I4,0)
2310         FOR I6 = 1 TO V
2320             E(I6) = B(I1,I2,I4,I6)
2330         NEXT I6
2340         GOSUB 1150
2350     NEXT I4
2360     GOSUB 1870
2370     NEXT I2
2380     NEXT I1
2390     CLOSE #2
2400     STOP
2410 'Matrix input subroutine
2420     INPUT "Matrix name"; FILE$
2430     INPUT "ROW,COL dimension"; ROW,COL
2440     INPUT "Number of polynomial variates"; V
2450     OPEN FILE$ FOR OUTPUT AS #1
2460     PRINT "Build a square diagonal matrix whose
2470     diagonal entries ";
2480     PRINT "are equal Y/N"; : INPUT OPT1$
2490     IF ((OPT1$ <> "Y") AND (OPT1$ <> "N")) THEN 2460
2500     PRINT CHR$(7); "**** COEF EXP1 EXP2 ... EXPn ****"
2510     PRINT " a semicolon ; represents end of polynomial"
2520     PRINT #1, ROW; COL; V
2530     FOR I = 1 TO ROW
2540         FOR J = 1 TO COL
2550             S = 0 : A$ = ""
2560             IF ((OPT1$ = "N") OR ((I=1) AND
2570             (J=1))) THEN 2610
2580             IF I = J THEN S = SS : PRINT #1,S :
2590             GOTO 2680
2600             PRINT #1,1;
2610             FOR K = 0 TO V : PRINT #1,0; : NEXT K
2620             PRINT #1,""
2630             GOTO 2750
2640             PRINT "("; STR$(I); ","; STR$(J); "): "
2650             WHILE (RIGHT$(A$,1) <> ";")
2660                 S = S + 1
2670                 INPUT A$
2680                 B$(S) = A$ : BB$(S) = A$
2690             WEND
2700             PRINT #1,S : SS = S
2710             FOR K = 1 TO S-1
2720                 IF OPT1$="Y" THEN B$(K)=BB$(K)

```

```

2700             PRINT #1,B$(K)
2710             NEXT K
2720             IF OPT1$ = "Y" THEN B$(S) = BB$(S)
2730             PRINT #1,LEFT$(B$(S),LEN(B$(S))-1)
2740             PRINT
2750             NEXT J
2760         NEXT I
2770     CLOSE #1
2780     GOTO 420
2790     'subroutine views a matrix file
2800     INPUT "Screen output Y/N"; OPT1$
2810     IF ((OPT1$ <> "Y") AND (OPT1$ <> "N")) THEN 2800
2820     INPUT "Matrix to view"; FILE$
2830     OPEN FILE$ FOR INPUT AS #1
2840     INPUT #1, ROW,COL,V
2850     IF OPT1$ = "Y" THEN 2890
2860     LPRINT "MATRIX "; FILE$; " IS "; STR$(ROW);
2870     " X"; STR$(COL);
2880     LPRINT " and has "; STR$(V); " polynomial variates"
2890     GOTO 2920
2900     PRINT "MATRIX "; FILE$; " IS "; STR$(ROW); " X";
2910     STR$(COL);
2920     PRINT " and has "; STR$(V); " polynomial variates"
2930     PRINT
2940     FOR I = 1 TO ROW
2950     FOR J = 1 TO COL
2960     IF OPT1$ = "Y" THEN 2970
2970     LPRINT FILE$; "("; STR$(I); ","; STR$(J); "): "
2980     GOTO 2980
2990     PRINT FILE$; "("; STR$(I); ","; STR$(J); "): "
3000     INPUT #1,S
3010     FOR K = 1 TO S
3020     LINE INPUT #1, A$
3030     IF OPT1$ = "Y" THEN 3030
3040     LPRINT A$ : GOTO 3040
3050     PRINT A$
3060     NEXT K
3070     IF OPT1$ = "Y" THEN 3080
3080     LPRINT
3090     GOTO 3090
3100     PRINT
3110     NEXT J
3120     NEXT I
3130     CLOSE #1
3140     GOTO 420
3150     'Subroutine transposes a matrix
3160     INPUT "Matrix to transpose"; FILE$
3170     INPUT "Matrix to store transpose in"; FILED$
3180     OPEN FILE$ FOR INPUT AS #1
3190     INPUT #1, ROW,COL,V
3200     IF ((ROW > ROWS) OR (COL > COLS)) THEN 1620
3210     FOR I = 1 TO ROW
3220     FOR J = 1 TO COL

```

```

3210         INPUT #1, TERMS
3220         IF TERMS > TERM THEN 1680
3230         COEF(I,J) = TERMS
3240         FOR K = 1 TO TERMS
3250             FOR L = 0 TO V
3260                 INPUT #1, A(I,J,K,L)
3270             NEXT L
3280         NEXT K
3290     NEXT J
3300 NEXT I
3310     CLOSE #1
3320     OPEN FILE0$ FOR OUTPUT AS #1
3330     PRINT #1, COL; ROW; V
3340     FOR I = 1 TO COL
3350         FOR J = 1 TO ROW
3360             PRINT #1, COEF(J,I)
3370             FOR K = 1 TO COEF(J,I)
3380                 FOR L = 0 TO V
3390                     PRINT #1, A(J,I,K,L);
3400                 NEXT L
3410             PRINT #1, ""
3420         NEXT K
3430     NEXT J
3440 NEXT I
3450     CLOSE #1
3460     GOTO 420
3470     'Extracts U,M,V,N matrices
3480     INPUT "Source matrix of U,M,V,N"; FILE1$
3490     INPUT "Disk to output to A/B"; D$
3500     IF ((D$ <> "A") AND (D$ <> "B")) THEN 3490
3510     D$ = D$ + ":"
3520     PRINT CHR$(7); "*** WARNING: files will be
output to "; D$;
3530     PRINT "U,M,V,N *** ";
3540     INPUT "OK Y/N"; OPT$
3550     IF OPT$ <> "Y" THEN STOP
3560     FILE2$ = D$ + "U"
3570     FILE3$ = D$ + "M"
3580     FILE4$ = D$ + "V"
3590     FILE5$ = D$ + "N"
3600     OPEN FILE1$ FOR INPUT AS #1
3610     OPEN FILE2$ FOR OUTPUT AS #2
3620     OPEN FILE3$ FOR OUTPUT AS #3
3630     OPEN FILE4$ FOR OUTPUT AS #4
3640     OPEN FILE5$ FOR OUTPUT AS #5
3650     INPUT #1, ROW, COL, V
3660     ROW1 = ROW/2 : COL1 = COL/2
3670     IF ((ROW1 <> INT(ROW1)) OR
(COL1 <> INT(COL1))) THEN 1760
3680     FOR I = 2 TO 5 : PRINT #I, ROW1, COL1, V : NEXT I
3690     FOR I = 1 TO ROW
3700         FOR J = 1 TO COL
3710             INPUT #1, S

```



```
3720      FIL = 5
3730      IF (I <= ROW1) THEN FIL = 2
3740      IF ((FIL = 2) AND (J > COL1)) THEN FIL = 3
3750      IF ((FIL = 5) AND (J <= COL1)) THEN FIL = 4
3760      PRINT #FIL,S
3770      FOR K = 1 TO S
3780          LINE INPUT #1,A$
3790          PRINT #FIL,A$
3800      NEXT K
3810      NEXT J
3820  NEXT I
```

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This paper seeks the solutions to a system of equations (equalities) in n variables by expressing the system in matrix algebraic form. Properties of the solutions to the ensuing matrix equation are investigated using similarity transformations. The three types of matrix equations to be studied are the linear equation $AX = b$, the Lyapunov equation $AX - XB = C$, the second-order Riccati equation $XDX + AX + XB + C = 0$, and the third-order Riccati equation $XAXBX + XCX + DX + XE + F = 0$. The entries of all matrices, including the solution X , are restricted to being polynomials in Γ having complex coefficients, where Γ is the n -tuple of indeterminates. That is, all matrices are elements of the ring

$$C^{m \times n}(\Gamma)$$

for m and n of appropriate size.

Because adding and multiplying matrices (having multivariate polynomial entries) is tedious in practice, an interactive BASIC program is presented in the appendix. This program, which can be used on a personal computer, permits the user to perform operations on matrices having multivariate polynomial entries. Via menu selections, the user may perform

- weighted addition between two matrices
- multiplication between two matrices
- create matrices, with an option of building a diagonal matrix whose diagonal entries are all equal
- view matrices
- transpose a matrix
- extract special submatrices (U,M,V,N of Chapter IV) from a given matrix

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